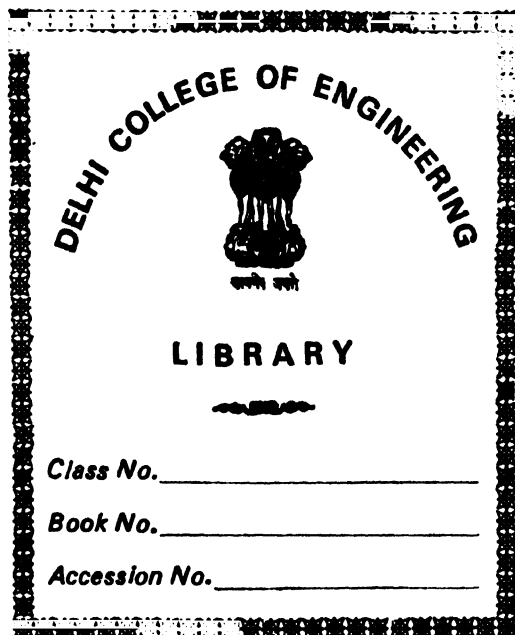




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Grateful acknowledgment is also expressed for the invaluable cooperation of the foremost manufacturers and engineering firms in making these volumes thoroughly representative of the best and latest practice in machine and pattern shops, foundries, and drafting rooms, and in the construction and operation of machine tools, and other classes of modern machinery; also for the valuable drawings and data.

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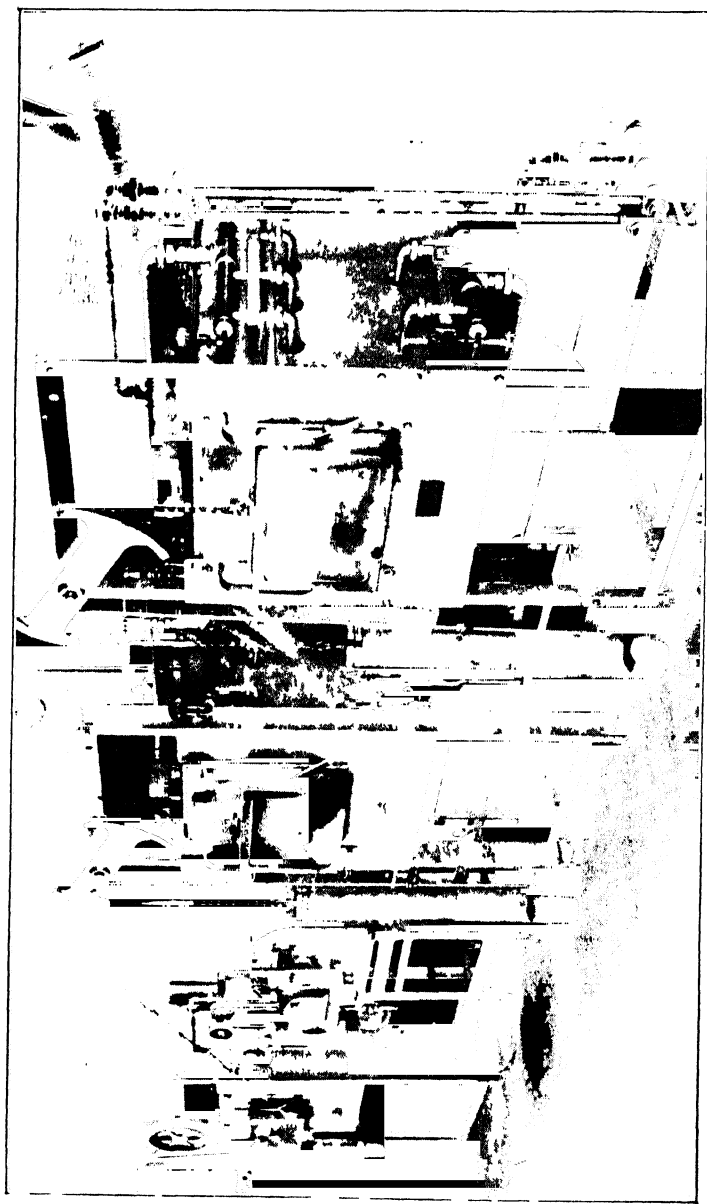
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EQUIPMENT FOR HEAT TREATING HIGH SPEED STEEL

Article of American Gas Furnace Elizabeth N J

F O R E W O R D

THE factory had its beginning a little more than a century ago with the development of water and steam power. Along with the development of machines came the need of mechanics to run these machines, to process the parts, and to assemble them into the finished articles. The evolution of both machines and mechanics has been marvelous, the accuracy of workmanship today being within limits not even measurable a century ago.

Production methods have increased enormously the output of our shops, and the machines which have made this improvement possible are of a diversified character—turret and other lathes, planers, multiple drillers, grinders, milling machines, presses, and the jigs, tools, and dies which go with them. All of these have contributed to the accuracy and speed of manufacture. The demands of the automobile industry have done wonders in hastening this development as the manufacture of parts in duplicate is absolutely necessary to maintain or lower the price of the assembled machines in the face of increased costs of labor and materials and the higher standards of manufacture. The fact that many of the present-day automobiles are shipped “knocked down” to assembly plants without ever having been fitted together is an eloquent testimonial to the accuracy with which duplicate parts are manufactured. Another contributing factor in modern production methods is the development of high-speed alloys for tools which enable the operators to run the machines at speeds hitherto unattainable.

And yet with all this wonderful development of the machines themselves and the design of what are termed “automatics,” the workman has not lost his skill. In fact, one trip

to a well-organized machine shop will teach any skeptic that the intelligent workman who has contributed so largely to the mechanical developments of the past years is more skilled, more intelligent, certainly better paid, and more interested in his work than ever.

But this same skilled mechanic is today a specialist. He has no opportunity to build a complete machine or even a small part of one; his active work is carried on along rather narrow lines. Consequently, it is all the more necessary for him to have a standard reading and reference work to help him in those shop lines with which he is unfamiliar, but of which he must have an understanding before he can fully master his own job and start toward the job ahead. Otherwise he becomes a machine, not knowing why nor what he is doing.

Modern Shop Practice is such a work—one which has been tested through several editions. The tremendous developments within the past few years have necessitated a thorough revision of many of the sections and a complete rewriting of others. Well-known teachers and practical men—authorities in their fields—have brought this material up-to-date in subject matter and illustrations. The authors have at all times kept in mind the practical nature of their subjects and numerous shop “kinks” and other helpful suggestions have been introduced. It is the hope of the publishers that this revised edition will continue to supply the needs of both the skilled mechanic and also the layman who is interested in mechanical affairs, as the former editions have done.

Grateful acknowledgment is given to the authors and collaborators—engineers and designers of wide practical experience and teachers of recognized ability and the leading manufacturers—without whose cooperation this work would have been impossible.

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†1 or professional standing of authors, see list of Authors and Collaborators at front of volume



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BLUEPRINT READING—Part I

—For the Machine Trades—

Definition of a Blueprint. A blueprint is a drawing which completely instructs the workman so that he is able actually to make in the shop the object which the drawing represents; in other words, a blueprint conveys to the mechanic all the information necessary to make the object. The student should constantly keep before him the idea that the workman must take a blueprint, and without any further instructions, verbal or written, produce the object as the draftsman intended it to be made. The instruction supplied by the blueprint should not only cover the form and size of the object, but also the kind of material of which it is to be made, the number of pieces desired, and the finish of its surfaces. A blueprint, therefore, is a sort of abbreviated language or shorthand method for conveying an amount of exact detail information which it would take many pages of manuscript to convey.

Blueprinting. The tracing made by the draftsman, cannot be sent into the shop for the workmen to use, as it would soon become soiled and in time destroyed, so that it is necessary to have some cheap and rapid means of making copies from it. Almost all drafting rooms have a frame for the purpose of making blueprints. These frames are made in many styles—some simple, some elaborate. A simple and efficient form is a flat surface, usually of wood, covered with a padding of soft material, such as felting. To this is hinged the cover, which consists of a frame similar to a picture frame, in which is set a piece of clear glass. The whole is either mounted on a track or on some sort of a swinging arm, so that it may readily be run in and out of a window. Figs. 1 and 2 are illustrations of a blueprint frame.

The print is made on paper, prepared for the purpose by having one of its surfaces coated with chemicals which are sensitive to sunlight. This coated paper, or blueprint paper,

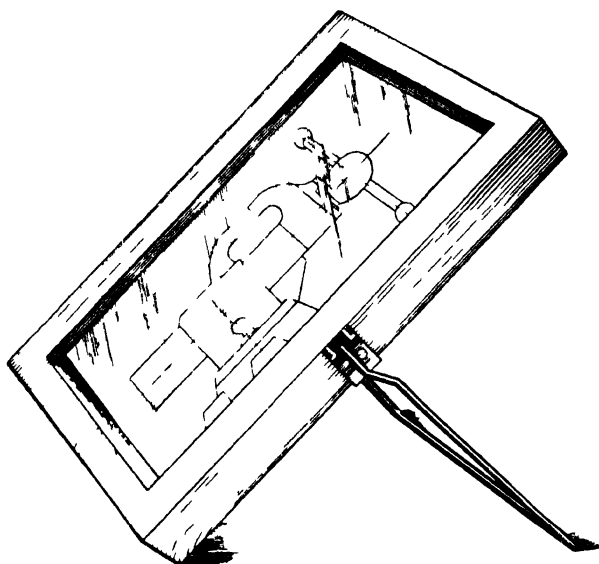


Fig 1 Blueprint frame Showing Tracing in It Ready to Be Printed

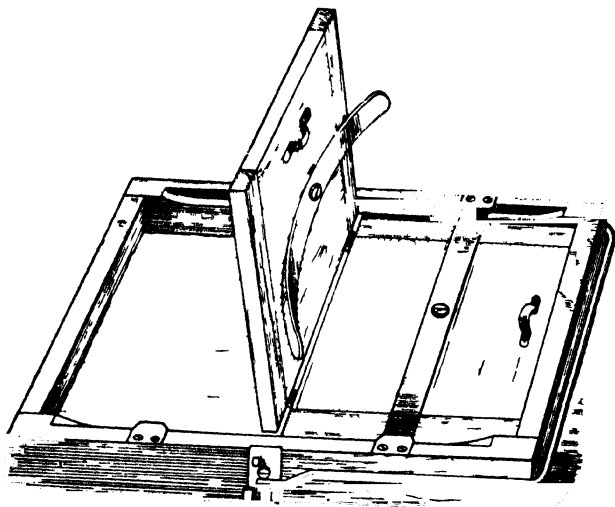


Fig 2 Back of Blueprint Frame Showing One Side Open

as it is called, is laid on the padded surface of the frame with its coated side uppermost; the tracing is laid over it right side up, and the glass pressed down firmly and fastened in place. Springs are frequently used to keep the paper or tracing against the glass. With some frames, however, it is more convenient to turn them over and remove the backs. In such cases the tracing is laid against the glass, face down; the coated paper is then placed on it with the coated side against the tracing cloth.

When the drawing has remained in the sun for a few minutes, the blueprint paper is taken out and thoroughly washed in clean water for several minutes and then hung up to dry. If the paper has been recently prepared and the exposure properly timed, the coated surface of the paper will now be of a clear deep blue color, except where it was covered by the ink lines, where it will be perfectly white.

Before the paper was exposed to the light, the coating was of a pale yellow color; if it had then been put in water, the coating would have all washed off, leaving the paper white. In other words, before being exposed to the sunlight the coating was soluble. But when the light penetrated the transparent tracing cloth and acted upon the chemicals of the coating, it changed their nature so that they became insoluble; that is, when put in water, the coating, instead of being washed off, merely turned blue. However, the light could not penetrate the ink with which the lines, figures, etc., were drawn, consequently, the coating under these was not acted upon and it was washed off when the paper was put in water, leaving a white copy of the ink drawing on a blue background. If running water cannot be used, the paper must be washed in a sufficient number of changes until the water is clear. It is a good plan to arrange a tank having an overflow, so that the water may remain at a depth of about 3 or 4 inches.

The length of time blueprint paper should be exposed to the light depends upon the quality and freshness of the paper, the chemicals used, and the brightness of the light. Some paper is prepared so that an exposure of one minute, or even

less, in bright sunlight will give a good print, and the time ranges from this to twenty minutes or more, according to the proportions of the various chemicals in the coating. If the full strength of the sunlight does not strike the paper, as, for instance, if clouds partly cover the sun, the time of exposure must be lengthened.

Blueprint paper should not be exposed to bright actinic light except during the process of printing. It is not, however, so sensitive as the ordinary dry plate and may be handled in a subdued light if the exposure is very brief. When not in use, the paper must be kept in a dry, dark place and should be hermetically sealed.

A more modern type of blueprint machine is the electric machine, usually arranged in the form of a vertical cylinder of glass, around which is placed the sensitized paper and the tracing, and along the longitudinal axis of which travels an electric arc lamp at a uniform speed. This speed is so adjusted that the right exposure is given to the sensitized paper. There are several other types of electric blueprinting machines, all based upon the same idea of uniform exposure of the sensitized paper to the rays of one or more arc lamps. These machines are a positive necessity to modern drawing offices because of the uncertainty of sunlight and the consequent limited capacity for turning out prints. With the electric machines there is no limit, and they may be run to their full capacity twenty-four hours a day if desired.

Preparing Blueprint Paper. Blueprint paper may be prepared as follows: Dissolve one ounce of ammonium citrate of iron in six ounces of water, and in a separate bottle dissolve the same quantity of potassium ferro-cyanide in six ounces of water. Keep the two solutions separate and in a dark place or in opaque bottles. To prepare the paper, mix equal quantities of the two solutions and with a sponge spread it evenly over the surface. Let the paper remain in a horizontal position until the chemicals have set on the surface. This will take only a few minutes, after which the paper is hung up to dry. It is necessary to darken the room by pulling down the shades

when the paper is prepared, as direct daylight would affect the sensitized surface. The prepared or sensitized paper must be kept in a closed drawer well covered with heavy paper, so that it will not come in direct contact with the light. Blueprint paper, however, is seldom sensitized at the present time by the users, because the manufactured blueprint paper can be obtained at a price which hardly warrants the trouble of sensitizing small quantities.

PRINCIPLES OF THE WORKING DRAWING

On the left of Fig. 3 is a sketch of an ordinary solid block. One side of the block is labeled "front" and another side "top."

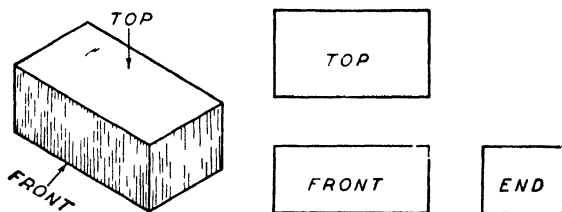


Fig. 3

To make a drawing of this block you would look first directly at the side which is labeled "top," then at the side labeled "front," and then at the right end.

The drawing labeled "top" is what you see when you look down directly on the top of the block and in doing this the sides, the bottom, and the ends are cut from your view.

The drawing labeled "front," which is at the right of the sketch of the block, is what you see when the block is directly in front of you. If you hold a rectangular object similar to this one directly in front of you, you will note that you are unable to see the top, bottom, back, or the two ends.

The drawing labeled "end" is what you see when you turn the end of the block toward you. This cuts from your view the top, the bottom, the two sides, and the left end.

The length of the block is shown by the horizontal lines in the top and front views. The width of the block is shown

by the vertical lines of the top view and the horizontal lines of the end view. The height is shown by the vertical lines of both the front and the end views.

These three drawings—the top, front, and end views—tell you all about the shape of the block, but they do not give the dimensions, which would tell you the actual length, the width, and the height of the block.

You will note that the mechanical drawings are drawn in bold solid lines. The solid line, which is common in all drawings, is used to represent the edges of an object which are seen.

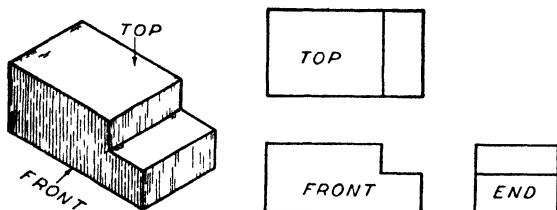


Fig. 4

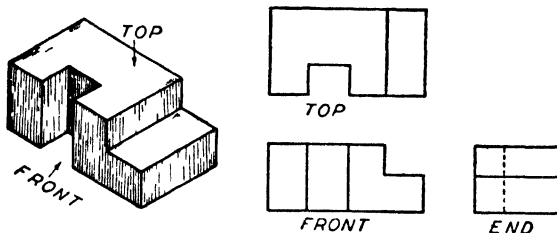


Fig. 5

The only difference between Fig. 3 and Fig. 4 is that a small step or recess is cut in the block in Fig. 4. Note how this fact is shown in the front view. In the top view observe how the edge of the step is shown by a solid line, because that is what we see as we look down on the figure. The end view shows three horizontal lines representing the edges of the top, the bottom, and the step, respectively. All the lines are solid because all the edges are seen.

All seen edges in drawing are represented by the solid line.

The difference between Fig. 5 and Fig. 4 is that a slot or keyway is added in the front view. This slot is shown in the front view and in the top view by solid lines. The top view shows both the width and the depth of the slot. In looking directly at the front of the object you will realize that the depth of the slot could not be shown in the front view.

In the end view, a dotted line is used to show an invisible edge. In looking at the end view, we cannot see the depth of the slot in the front view and so its position is shown by a dotted line.

Note that this dotted line appears at the left of the end view, rather than at the right, its position in the drawing coinciding with the position of the slot as shown in the front and top views, and therefore it is correct.

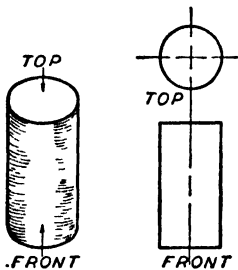


Fig. 6

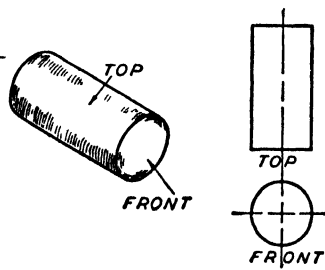


Fig. 7

All unseen edges in drawings are represented by the dotted line.

Fig. 6 shows a sketch and a mechanical drawing of a cylinder or a round object which is cut off square at each end. Note the parts which are labeled "front" and "top" in the sketch.

Look at the mechanical drawing of the cylinder in the upright position and note that the top view is a circle and the front view a rectangle. A cylinder always appears as a rectangle when you hold it directly in front of you in this position. You do not see the top and the bottom, and therefore these edges appear as straight lines. The top view which shows as a circle tells you that the figure is round, rather than square or

any other shape. Therefore, in looking at this mechanical drawing with a front view which shows as a rectangle and a complete top view which shows as a circle, we know that the object is a cylinder.

Fig. 7 shows the cylinder lying on its side, and this fact changes the position of the front and top views. In comparing the two mechanical drawings in Figs. 6 and 7, it will be noted that the facts concerning the shape of the object have not been changed although the views are reversed. We have one view showing a rectangle and the other view showing the circle, which again tells us that the object which we are looking at must be a cylinder.

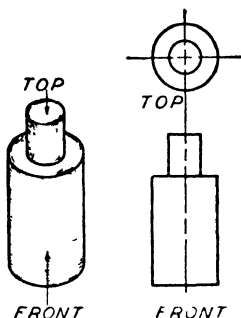


Fig. 8

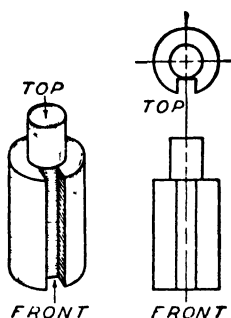


Fig. 9

In Figs. 6 and 7 a line, known as a center line, is used. A center line is a straight line broken into long and short dashes. It is used in all drawings where circles are the base of the drawing or where dimensions are taken from the center of the object.

Fig. 8 is very similar to Fig. 7, with an additional feature in shape, for the cylinder now has a peg at one end. The length and width of this peg are shown in the front view of the mechanical drawing. Note how the two circles in the top view of the mechanical drawing represent, first, the outside edge of the cylinder and, second, the edge of the peg. These two views in the mechanical drawing of Fig. 8 tell us clearly that the figure we are looking at is a cylinder with a round peg at one end. Note again the use of the center line.

Fig. 9 is similar to Fig. 8 with a construction detail in the way of a slot or keyway made in the front of the cylinder. In the mechanical drawing note how the width of this slot or keyway is shown by the two solid lines in the front view and how both the width and the depth of the keyway are shown in the top view by the notch in the outside circle, because this indicates quite clearly where the slot or keyway is placed.

If we apply the correct terms to these different views, we say plan view for the top view, front elevation for the front view, and side elevation for the end or side view.

From the foregoing we can form the following rule. What you see in the plan or top view is what you see of the object as you look down on it from above, and any dotted lines in that view indicate hidden edges, corners or surfaces, in or on the object as viewed from that position.

What you see in the elevation or front view is what you see of the object as you look directly at it from the front, and any dotted lines in that view indicate hidden edges, corners or surfaces, in or upon the object as viewed from that position.

What you see in the side elevation or end view is what you see of the object when you look at it from the side or end, and any dotted lines in that view indicate hidden edges, corners or surfaces, in or upon the object as viewed from that position.

The plan view is always placed above and directly in line with the elevation view. The elevation view is always below and directly in line with the plan view. The end view is usually placed to the right or left of the elevation view and directly in line with it, but it will sometimes be found in line with the plan view.

We have seen on previous pages that a drawing or blueprint is a plan from which the workman makes the object. We do not have the object to look at as a general rule. The object is made from the drawing. To visualize or to construct a mental picture of the object, we must study or read the blueprints. A good mechanic who can read a blueprint will be able to visualize the finished article from the study of that blueprint. For instance a pattern maker will visualize the finished pattern before he actually starts its construction.

To find out the shape of the object, you start with the elevation. By studying that view it will enable you to visualize the shape of the front of the object, the plan view, the top of the object, etc. The draftsman has a mental picture of the object he is going to design or lay out and puts this down on paper. He is guided in the choice of views, as to which side of the object should be used as a front view or a top view, by the position the object actually takes in a complete assembly, so that the mechanic knows that when a study of the front view is made, as shown on the drawing, he is actually considering the front view of the part or object.

In simple construction, such as shown in Figs. 3 to 9, it is easy to draw up in the mind the shape of the object from the two or three views shown, but in complicated construction it is often necessary to take a particular part of one view and follow it all through the views shown, and get a mental picture of that part, then take another part, etc. By doing this a mental picture gradually can be built up and a complete picture formed of the construction required.

It is excellent practice for the student to practice this power of visualization by calling to mind objects that he sees around him every day. Start with simple objects around your own home, such as a table or bookcase, study them carefully from all sides, then close your eyes and see if you can bring to mind a picture of the construction of the different sides, also locate any interior parts and decide where the hidden edges come and how they should be shown. It is a good plan to turn away from the object you are trying to visualize and make sketches on paper from the mental picture you form, with as many details as you can remember, carefully put in. Keep the views of the object you are putting on the paper in the correct position relative to each other, as previously explained, and show hidden parts in dotted lines in the respective views. Edges of round objects appear as straight lines when looking directly at them, and are shown as such in the view for that position. The top of a round tin can or the edge of a phonograph record when seen on edge appear as straight lines.

RELATION BETWEEN OBJECTS AND MECHANICAL DRAWINGS

In Fig. 10 at 1 the projection on the block is shown by two full lines in the front view of the mechanical drawing, and the actual shape of the projection is shown on the top view. In this case the two drawings give all the facts concerning the object.

At 2 the series of vertical lines in the front view represent the edges of the various parts of the V block. Note, however, that it takes the top view to complete all the facts regarding these lines. The middle line in the front view represents the corner in the bottom of the V. Two views give all the facts concerning the object.

At 3 note the center line because a circle is introduced. The front view alone would not give you the fact that the piece which projects beyond the base is circular in shape.

At 4 the two dotted lines in the front view show that the hole goes all the way through the wrench.

At 5 the front view shows the sides of the cylinder as extending the full length, even though it appears broken by the cross-shaped piece which is on the top. This means that the front view is taken with the cross at right angles to the observer. Note again the introduction of the center line.

At 6 the front view shows the shape of the triangular piece on the rectangular block, but it requires the top view to show how far back it cuts into the block.

At 7 the front view shows the thickness of the hexagon, and the top view shows its true shape.

At 8 the top view shows the shape of the angle, and the front view shows that it is hollow, as indicated by the dotted lines. The dotted lines also show the thickness of the sides of the angle. Study this drawing carefully to see clearly the necessity of the dotted lines.

At 9 the front view shows the height of the projecting lug on the top of the round base. The top view shows that the base is round and that the lug is round. Note the similarity of the top view at 10 to the top view at 9, and yet the front view shows that the shapes of the figures are entirely different.

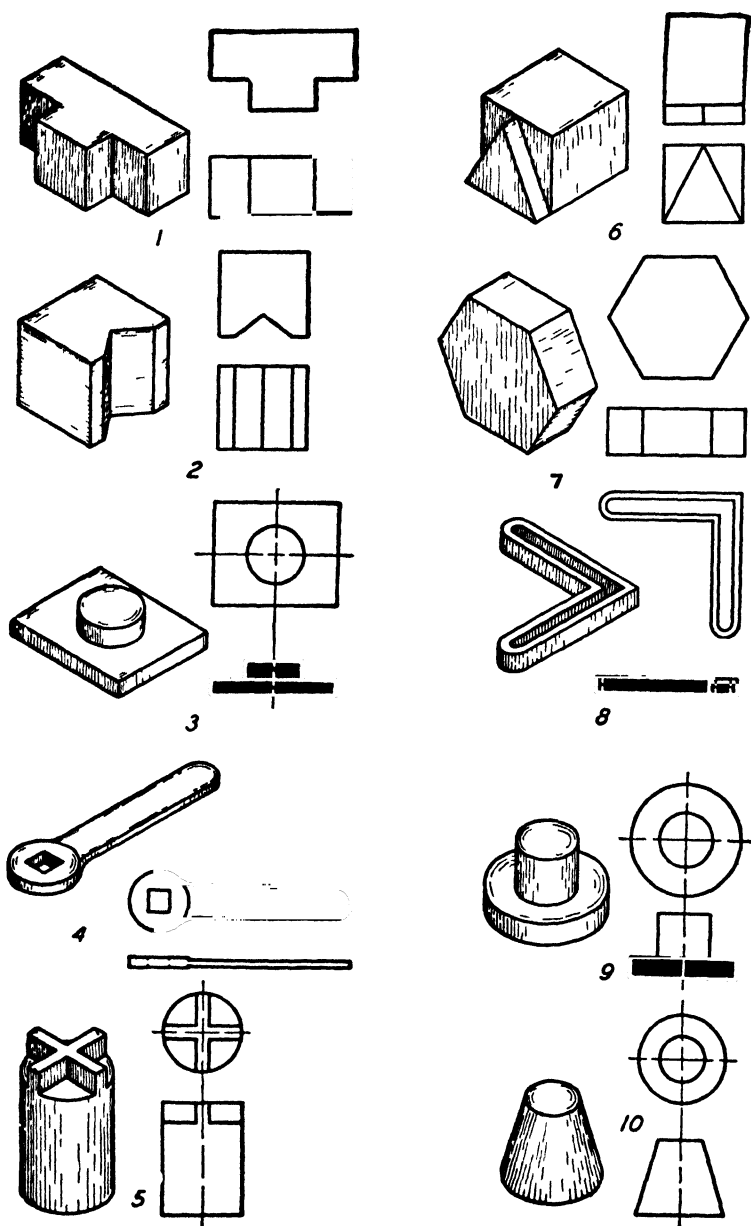


Fig 10

As an exercise in reading these drawings, it would be worth while to make a rough copy of each of the pictures of the object and then make an effort to sketch a simple freehand working drawing of the object and check your result with the drawings which are shown.

Definitions for Some of the Most Commonly Used Terms in the Machine Shop

Before studying the working drawing, one should become familiar with the following terms:

Drill. A circular tool used for machining a hole. Plate I, Fig. F.

Counterbore. A tool that enlarges to a certain depth a round hole already machined. The pilot of the tool fits in the smaller hole and the larger part counterbores or makes larger the end of the hole. Plate I, Fig. D.

Tap. A tool that cuts or machines threads in the side of a round hole. Plate I, Fig. H.

Reamer. A finishing tool that finishes a circular hole more accurately than a drill. Plate I, Fig. G.

Fillet. The rounded corner or portion which joins two surfaces which are at an angle to each other. Plate I, Fig. A.

Spot-face. The machining of a flat surface around the end of a hole to allow a bolt-head or nut to seat squarely on the surface and is a little larger in diameter than the distance across the corners of the bolt-head or nut. Plate I, Fig. C.

Eccentric. Where two circular parts are in one rotating unit and yet have different centers, one is said to be eccentric to the other. The amount of eccentricity is the radial distance between the two centers. Plate I, Fig. E.

Boss. Usually a round projecting surface on another surface. Plate I, Fig. C.

Face of a Gear. That portion of the tooth curve above the

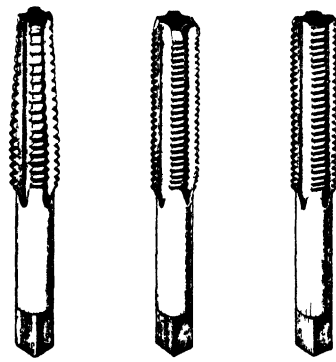
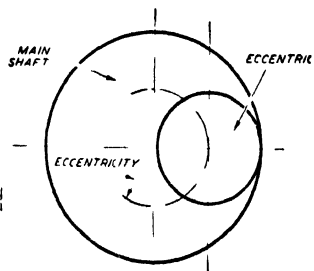
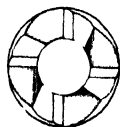
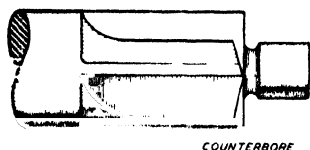
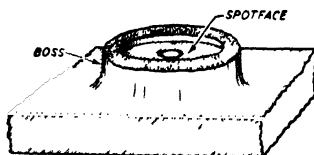
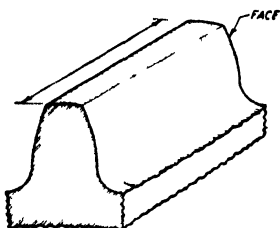
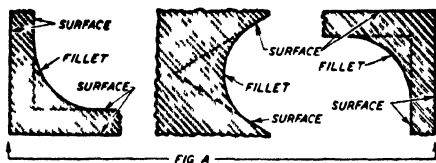


Plate I

pitch circle and measured across the rim of the gear from one end of the tooth to the other Plate I, Fig B

Core. A core is used for forming a hole or cavity in a casting by placing sand cores in impression in the mold which are left by core prints which, in turn, are attached to the pattern

Cored hole. At the time of casting a sand core is so placed in the mold that the metal flows around it When the casting is cold, the sand core is broken away leaving the hole in the casting

Bore. A hole machined in a piece that is made with a tool that is known as a boring tool

Face. An operation that machines the sides or ends of the piece

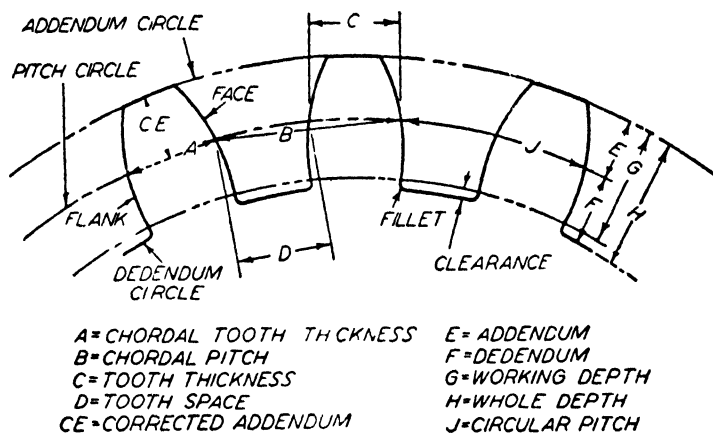


Plate II

F. A. O. This symbol on a drawing means that the piece is machined or finished all over

C. I. This symbol is the abbreviation for indicating cast iron

Note: The inch mark on a working drawing is written thus—" For example three inches is written 3"

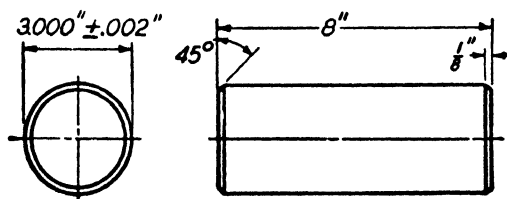
The mark for feet is written thus—" For example seven feet three inches is written 7' 3"

Many blueprints pertaining to gears are used in machine shop

work, as well as in the drafting room. It is well for the student to know the names or terms used in conjunction with gear construction. Plate II shows the names or terms used for different parts of the gear and its teeth.

READING WORKING DRAWINGS

In reading the following typical working drawings—some consist of two views, others of three views, and others are sectional drawings—the student is given an opportunity to learn just how different mechanics use blueprints.



F.A.O.

Fig. 11 Working Drawing



Fig. 12. Sketch of Cold Rolled Steel Pin

Cold-Rolled Steel Pin. Fig. 11 shows an end view and a front view of a working drawing of a pin that is $3.000'' \pm .002''$ in diameter and $8''$ long. The ends of the pin are chamfered or beveled off. This chamfer is $\frac{1}{8}''$ from the ends of the pin and at an angle of 45° .

The dimension $3.000'' \pm .002''$ indicates that an oversize and undersize limit of $.002''$ is allowed for the diameter. If the piece is machined $.002''$ larger or $.002''$ smaller than $3.000''$, it would be considered satisfactory. The symbol F.A.O. means that this piece is machined or finished all over.

A sketch of the pin is shown in Fig. 12, which will help the student to get a better understanding of the working drawing.

QUIZ QUESTIONS*

1. What is the diameter of the pin?
2. What is the angle of the chamfers?
3. What is the distance from the ends of the pin to the ends of the chamfers?

*No written answers are required for Quiz Questions.

Brass Ring. Fig. 13 shows a working drawing of a brass ring that is 3" in diameter and $\frac{1}{8}$ " thick. The thickness of the ring is given on the end view. The large hole in the ring is 2" in diameter. This dimension is found on the front view. There are four small holes drilled through the ring that are

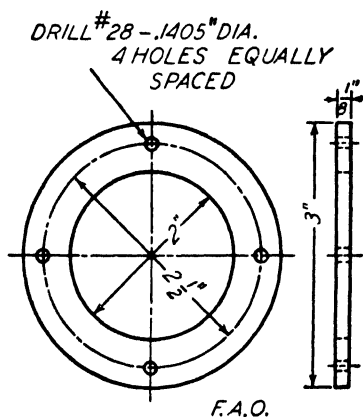


Fig. 13. Working Drawing

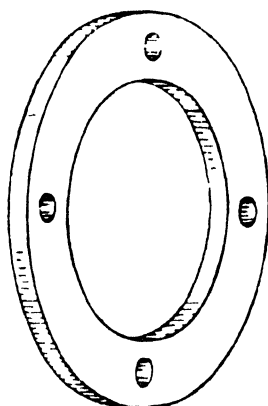


Fig. 14. Sketch of Brass Ring

equally spaced. The size of these holes is indicated directly above the drawing, which is drill No. 28—.1405" in diameter. These holes are located on a circle that is $2\frac{1}{2}$ " in diameter. If we wish to know whether or not these holes are drilled all the way through the ring, we look at the end view and find that they are, because dotted lines are drawn from one side of the view to the other.

The symbol F.A.O. means that this piece is machined or finished all over.

A sketch of the ring is shown in Fig. 14. This sketch will help the student to have a better understanding of the working drawing.

QUIZ QUESTIONS*

1. What is the diameter of the circle on which the holes are to be drilled when drilling the ring?
2. What size drill should be used?
3. How thick is the ring?

*No written answers are required for Quiz Questions.

Rectangular Steel Cover. Fig. 15 shows a front and end view of a working drawing of a rectangular steel cover. The front view shows that the over-all dimension of this cover or plate is 10 inches long and 6 inches wide; that the large hole is in the center and is 5 inches from the end and 3 inches from the side. The size of the hole is indicated by the size of drill that is used for making the hole, which is $1\frac{1}{2}$ inches in diameter.

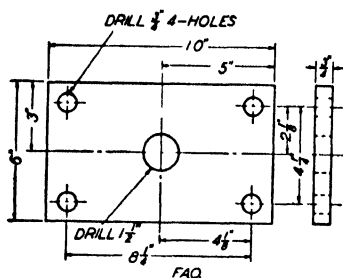


Fig 15 Working Drawing of a Rectangular Steel Cover

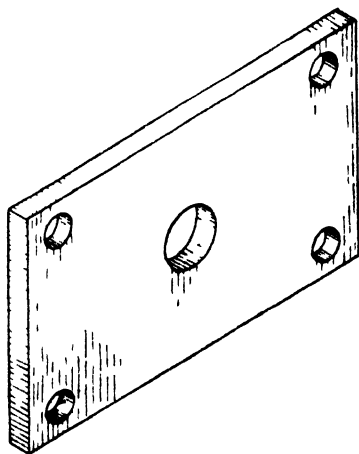


Fig 16 Sketch of a Rectangle Steel Cover

Note how the four small holes which are shown at the corners are located in relation to the center lines. These holes are $4\frac{1}{8}$ inches from one center line and $2\frac{1}{8}$ from the other center line. The distance between the centers of these holes is shown as $8\frac{1}{4}$ inches and $4\frac{1}{4}$ inches.

All that is necessary to get from the end view is the thickness of the plate, which is shown as $\frac{3}{4}$ inch. The dotted lines in the end view show that each of the five holes is drilled through the plate, also the positions of these holes.

The symbol F.A.O. means that this piece is machined or finished all over.

Fig. 16 shows a sketch of the rectangular steel cover. This sketch will help the student to have a better understanding of the working drawing.

QUIZ QUESTIONS*

1. What is the size of the center hole?
2. What is the size of the four small holes?
3. Are the holes drilled through the piece?
4. How are the centers for the small holes located?

*No written answers are required for Quiz Questions.

Round Brass Cover. Fig. 17 shows a working drawing of a round brass cover for a pump body. The two standard views are given—a front view with the cover standing on its edge, and a side view in the same position.

The outside diameter of this round cover is $5\frac{1}{2}$ inches. This dimension is placed between the two views.

The next important dimension is the dimension of the circle which locates the centers of the holes which are to be drilled in the cover. This is 5 inches in diameter and is shown on the front view. The location of the holes is also shown on the front view, the outer circle for the holes being that part which is counterbored and the inner circle is the drilled hole which goes clear through the cover. These holes are spaced equally on the center line and are 60° apart. The dimension showing the $\frac{1}{8}$ " *R* indicates that the ribs are joined together with a $\frac{1}{8}$ " radius or fillet. The dotted circle in front view indicates the recess which is on the under side of the cover. The size of this recess is $4\frac{1}{2}$ inches in diameter and $\frac{1}{16}$ inch deep, as shown on the side view.

In the side view, the depth of the holes which are drilled and counterbored will be seen in dotted lines, the counterbored

part being shown about half way through the section of the cover and the drilled hole being shown all the way through. Note the projections of the holes in the side view.

In a drawing of this kind, the size of the screw is indicated instead of the size of the drill and the counterbore. Therefore, the machinist will have to use a drill and a counterbore to suit a No. 10—32x.190" diameter oval fillister head machine screw.

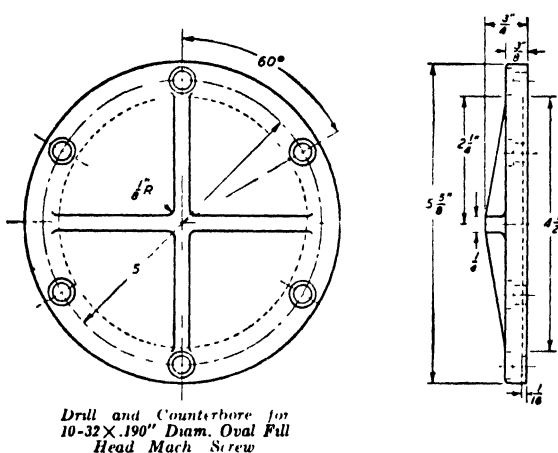


Fig 17. Working Drawing for a Round Brass Cover

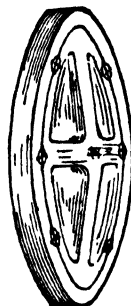


Fig 18.
Sketch of
Casting for a
Round Brass
Cover, Show
ing Layout for
Drilling

(An oval fillister head screw is a screw which is rounded or oval on the top and flat on the underside of the head.) No. 10 is the number of the screw; 32 indicates the number of threads per inch, then the pitch of the screw is $\frac{1}{32}$ " ; and .190" is the diameter of the shank of the screw.

In the side view the distance from the bottom of the cover to the top of the ribs is given as $\frac{3}{4}$ inch and the thickness or width of the ribs is given as $\frac{1}{4}$ inch. In taking the right-end view, the right-hand end of the front view is turned toward you, so that the front of the object is the left-hand side of the end view, while the back of the object is toward the right of the end view. It is well to remember this because it will help you in visualizing construction.

The symbol *f* means that this piece is machined or finished.

Fig. 18 shows a sketch of a casting for a round brass cover. This sketch will help the student to have a better understanding of the working drawing.

QUIZ QUESTIONS*

1. What is the outside diameter of the brass cover?
2. What is the diameter of the recess in the bottom of the cover?
3. What is the depth of the recess in the bottom of the cover?
4. What is the thickness of the ribs?
5. What is the distance from the top of the ribs to the bottom of the cover?

*No written answers are required for Quiz Questions.

Cast-Iron Crank. The working drawing of the cast-iron crank shown in Fig. 19 consists of a top view and a front view. As it is a little more difficult to visualize from this working drawing just how the finished piece would look, an isometric drawing, which might be likened to the finished piece part, is shown in Fig. 20. This piece part can be considered as a crank, having a $1\frac{1}{4}$ " hole through the one end and a 1" hole through the other end. In the end having the $1\frac{1}{4}$ " hole is a groove, which is called a keyway.

As the pattern maker would be the first mechanic to use the drawing for this particular piece, we shall consider the dimensions on the drawing that he would use.

Having procured the desired size of stock for the pattern, first, he will lay off the center lines in order to determine the distance between the centers of the $1\frac{1}{4}$ " and the 1" holes.

These centers are to be $5\frac{1}{4}$ " apart, and you will find this dimension between the top view and the front view.

Next, the pattern maker will want to know the diameter of the large hub, that is, the hub in which the $1\frac{1}{4}$ " hole is; and also the diameter of the small hub in which the 1" hole is.

You will find these dimensions above the front view. The diameter of the large hub is to be 2" and the diameter of the

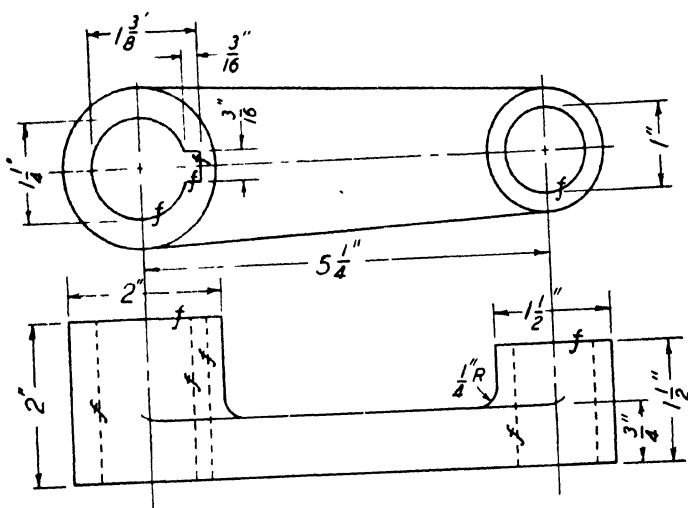


Fig 19 Working Drawing for a Cast-Iron Crank

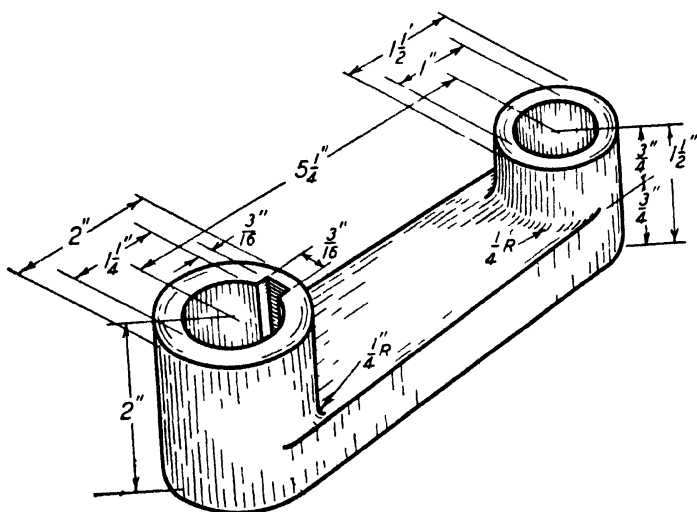


Fig 20 Sketch of a Cast-Iron Crank

small hub is to be $1\frac{1}{2}$ "; and the length of the large hub is to be 2" and the length of the small hub is to be $1\frac{1}{2}$ ".

The pattern maker must now determine whether the holes go entirely through the hubs and whether the ends of the hubs are to be machined.

If you will refer to the front view, you will find that the invisible or dotted lines that have been projected from the top view pass from one end of the hub to the other end. This indicates that the holes are to go all the way through the hub. On the upper end of the hubs in the front view is the symbol *f*, which indicates machining. This applies also to the holes through the hub.

The next mechanic that would use the drawing for this crank is the machinist or drill-press hand. His job would be to machine the $1\frac{1}{4}$ " and 1" holes and the ends of the hubs.

This particular machining processes do not strictly apply to the drill press hand. Such operations could be handled on other machines, such as the lathe or the milling machine.

QUIZ QUESTIONS*

1. What is the distance between hole centers?
2. What should be the diameter of the large hub?
3. What should be the length of the large hub?
4. What should be the width of the keyway?
5. What size of reamer should be used for the hole in the small hub?

***No written answers are required for Quiz Questions.**

Cast-Iron Bracket. Fig. 21 shows a front view and an end view of the working drawing of a bracket. A study of this working drawing, which has been made according to the principles of orthographic projection, and the isometric drawing, Fig. 22, will help you to read the more complicated working drawings with which mechanics frequently come in contact.

The milling machine operator as well as the all-around machinist use working drawings. Therefore, let us see just how this drawing, Fig. 21, would be applied to their particular jobs.

The end view of this drawing shows a $1\frac{1}{8}$ " reamed hole in the upper hub and $1\frac{3}{8}$ " reamed holes in the lower hubs. Referring to the front view, you will find that the ends of the hubs must be faced (machined).

Referring to the upper part of the bracket, you will find that in machining the left end of the hub it will be necessary to make the distance from the end of the hub to the rib $\frac{3}{16}$ "; and that the length of the hub is to be $4\frac{1}{4}$ ".

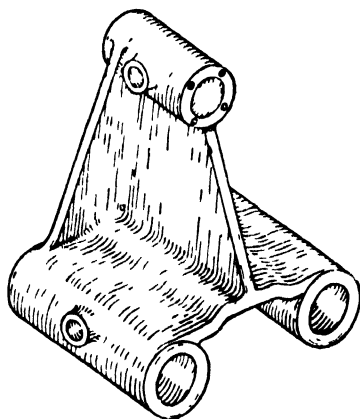


Fig. 22. Sketch of a Cast-Iron Bracket

The next step is to find the distance that the lower hub projects beyond the rib. Referring to the drawing, you will find this distance is to be $\frac{1}{2}$ ". The length of the lower hub is to be 6".

The layout man would want to know the distance between the centers of the holes in lower and upper hubs

The right side of the front view, shows the distance to be
 $8.000'' + .005''$
 $- .003''$.

Next, the layout man must find the distance from the center line to the center of the $1\frac{3}{8}$ " reamed hole in the two lower hubs; also the distance between the centers of the two $1\frac{3}{8}$ " reamed holes.

On the end view, you will find that the distance from the

center line to the center of the $1\frac{3}{8}$ " reamed hole is to be 3"; and between the centers of the two $1\frac{3}{8}$ " reamed holes is 6".

On the sides of the lower hub are bosses and through these bosses and hubs, holes are to be tapped. The layout man must find out the location of these tapped holes.

On the lower part of the front view, you will find the dimension $3\frac{3}{16}$ ", which is the distance the tapped holes are to be located from the end of the hub to the center of the hole.

Now the layout man must find the distance from the center of the $1\frac{3}{8}$ " holes to the ends of the bosses. Referring to the lower part of the end view, you will find that this distance is to be $1\frac{3}{8}$ ".

There are four holes which are to be drilled and tapped in the upper hub. These holes are not located on the center line of the upper hub and to find their location, angular dimensions are given. You will see that these holes are located at 45° from the center line of the hub.

QUIZ QUESTIONS*

1. To what length must the upper hub be faced?
2. What must be the distance between the centers of the $1\frac{3}{8}$ " reamed holes?
3. What must be the center distance between the $1\frac{1}{8}$ " and $1\frac{3}{8}$ " reamed holes?
4. What size drill would you use before reaming the $1\frac{1}{8}$ " hole?
5. What size drill would you use before tapping the $\frac{1}{2}$ " holes?

*No written answers are required for Quiz Questions.

Cast-Iron Box. Fig. 23 shows three views of the working drawing of a cast-iron box—a front view, a top view, and a right-side view. An isometric sketch of the box is shown in Fig. 24. This sketch, showing the cast-iron box clamped to the shaper table and the tool in contact with the piece part starting to cut the pad at the bottom of the box, will help the student to visualize just how the finished piece part will look.

In reading the drawing, the shaper hand finds that the base of the cast-iron box is to be machined, that is, the piece is to be machined off until the thickness of the base is of the proper dimension. This dimension is found on the front view. It is $1\frac{3}{8}$ ".

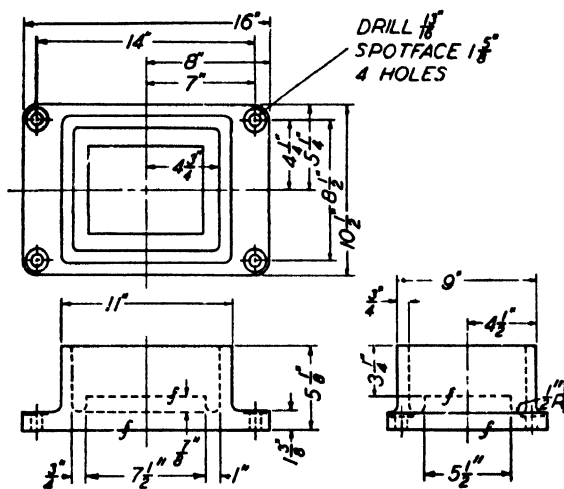


Fig. 23. Working Drawing of a Cast-Iron Box

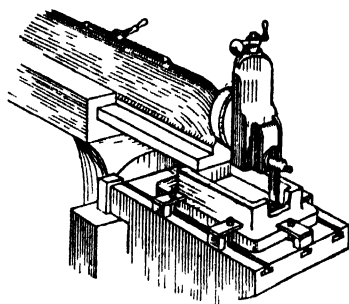


Fig. 24. Sketch Showing Box Being Machined on Shaper

Inside of this cast-iron box there is a pad $7\frac{1}{2}$ " long and $5\frac{1}{2}$ " wide. These dimensions are found on the front view and the end view. The height of this pad is to be $\frac{3}{8}$ ". This dimension is found on the front view. The distance from the top of the

pad to the top of the box is to be $3\frac{1}{4}$ ". This dimension is found on the end view.

This job must be completed on the drill press as there are four holes and four spotfacings (machining) required. In order to determine the proper size of drill to use and the size of the spotfacing tool, the drill press hand refers to the drawing and finds that the drill is to be $\frac{1}{8}$ " and the spotfacing tool is to be $1\frac{1}{8}$ ".

QUIZ QUESTIONS*

1. What is the distance from the top of the pad to the bottom of the recess?
2. Should the bottom of the box below the pad be machined?
3. What is the overall height of the box?
4. What is the thickness of the base?
5. What is the size of the spotface?
6. Do the four holes in the base of the box go through the base?
7. What is the size of the drill for these holes?
8. What is the distance between centers of the holes?

***No written answers are required for Quiz Questions.**

Cast-Iron Cylinder. Fig. 25 shows the working drawing of a cast-iron cylinder in three views—a front view, an end view, and a top view. Fig. 26 is a sketch of a casting for this cylinder.

This sketch will enable the student to visualize the construction and design of the cylinder.

To study the construction of the cylinder, you refer to the front view directly below the cylinder. You will find that the base is to be $\frac{7}{8}$ " thick and 12" wide. Above the base are two lines which taper up and make connection with the invisible circle, which is the outside diameter of the cylinder. Looking at the front view, it is hard to tell whether these are ribs or whether the tapered portion directly above the base is a solid mass of metal. So you refer to the right-side view and find that there are two ribs $\frac{3}{4}$ " thick that make connection from the base to the cylinder. This is more clearly shown in

the top view. Here you will find that there are not only two ribs on the sides of the cylinder, but that there is also a rib which runs lengthwise of the cylinder directly below the center of the cylinder.

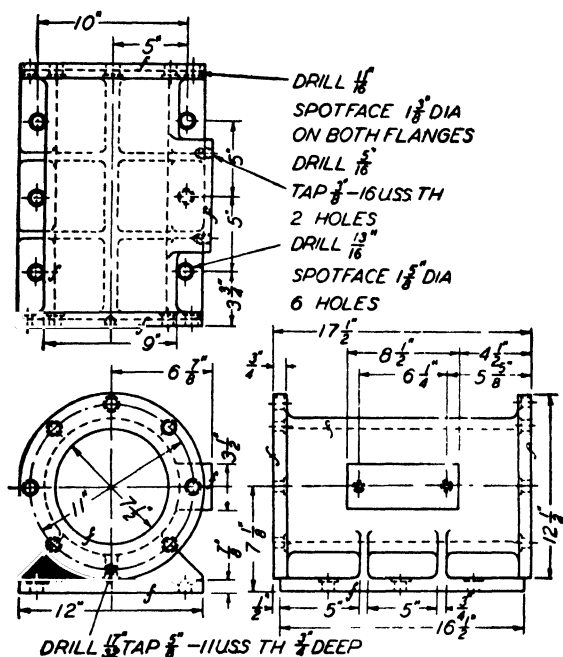


Fig. 25. Working Drawing of a Cast-Iron Cylinder

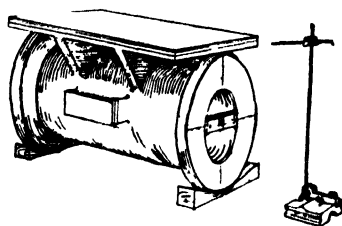


Fig. 26. Sketch Showing Casting Being Laid Out

In order to determine the thickness of the flanges on the ends of the cylinder, you refer to the side view. Here you will find that this thickness is to be 1 1/4", this dimension being shown directly above the flange on the left end of the cylinder.

Having set up the job on the horizontal boring mill, the operator will need to know the length of the cylinder. Sometimes you will find this dimension specified on the top view but in this particular case it is shown directly above the right-side view. The length is to be $17\frac{1}{2}$ ".

Before boring (machining), it is necessary for the mechanic to know what diameter to bore the cylinder. Referring to the front view, you will find inside of the circle indicating the hole the dimension $7\frac{1}{2}$ ". The symbol f on the circle indicates that the cylinder must be bored out.

QUIZ QUESTIONS*

1. To what size should the cylinder be bored?
2. What should be the distance from the center of the cylinder to the $3\frac{1}{2}$ " x $8\frac{1}{2}$ " pad?
3. To what length should the cylinder be machined?
4. How thick should the base be after it has been machined?
5. What is the distance from the bottom of the base to the center of the cylinder?

*No written answers are required for Quiz Questions.

Cross Sections or Cutaway Views. Sometimes the construction of an object is so complicated that it is difficult to show it in the regular projection views. If too many dotted lines are used to show the hidden edges in the object, the views become confusing and hard to read. To show the internal construction clearly, the object is imagined cut into sections as it could be in reality with a saw. A simple bushing is shown in Fig. 27 in plan and elevation views. It really would not be necessary to show a cross section of this, but it will serve as a means of illustration.

For the cross section of this simple bushing is supposed to be cut down through the center on the line AA . The arrows indicate the direction from which the section is viewed. Let the bushing stand in a vertical position, and then take the saw and cut through it downward, on the line indicated; if the portion nearest to the right of the center line is taken away, the remaining portion would be seen from the direction indicated by the arrows

and it would be like the section shown. The section itself is also indicated as section *AA*. It is seen that the saw would cut the walls of the hole only, and these are the surfaces which should be crosshatched. Therefore, surfaces or parts actually cut by the section line are the parts to be crosshatched.

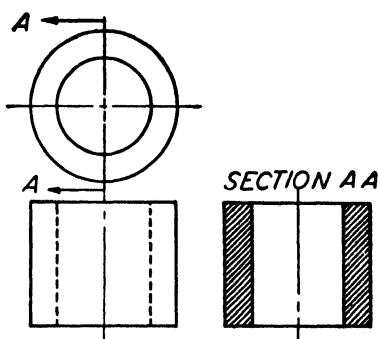


Fig. 27

In Fig. 28 is shown a sketch of a rectangular piece of metal with a piece cut out of one end, a round hole drilled through one side, and a rectangular hole through the other. Two sections are to be taken on lines *AA* and *BB*.

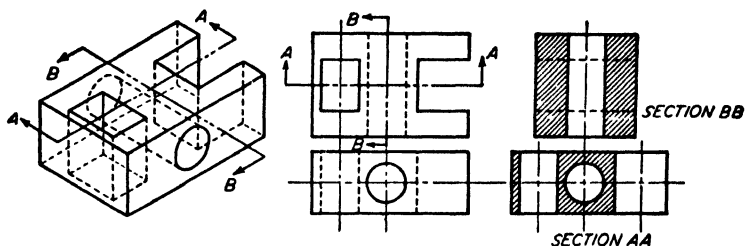


Fig. 28

The part is cut on line *AA* and the front portion removed, then the section shows the edges of the part cut at the left end. The two lines representing the square hole are at the left end; in the center is the circle which is the round hole; while the plain part is the right-hand end or portion which has a piece cut out of it which would be untouched by

the saw. This part untouched by the saw when cutting the object is not crosshatched.

The section *BB* is taken down through the center of the round hole and the portion left is viewed in the direction of the arrows. The remaining section will be shown as in section *BB*. The full lines show the edges of the round hole. The dotted lines show the hidden edges of the rectangular hole.

Fig. 29 shows a pulley. The section is taken as indicated and we see the edge of the rim of the pulley, while the pulley

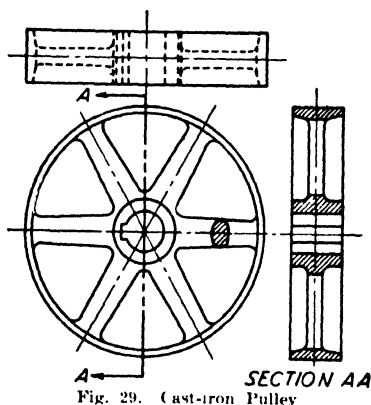


Fig. 29. Cast-iron Pulley

arm is left full and not crosshatched because it is not cut. The hub is cut and we see the corners of the keyway and walls of the hole, and the walls of the hole being cut are crosshatched. To show the shape of the pulley arm a small revolved section is shown in the arm, which indicates the section that would be seen if the arm was cut.

In Fig. 30 is shown a spring bracket which has a web to give strength without a large amount of weight. A section taken on the center line *AA* would also actually cut the web as well. It is standard drafting room practice not to crosshatch the rib, to avoid confusion, when the cut is taken in the same direction as the rib is located. If, on the other hand, the section was taken on the line *BB*, it would cut across the web and this section would be crosshatched.

In the circular portion, or hub, of the bracket a bronze bushing is inserted, and as this is of a different metal than the bracket itself, it will, of course, have a different crosshatching convention. When two different parts of the object come together and are crosshatched, the crosshatching lines change in direction, Figs. 30 and 31.

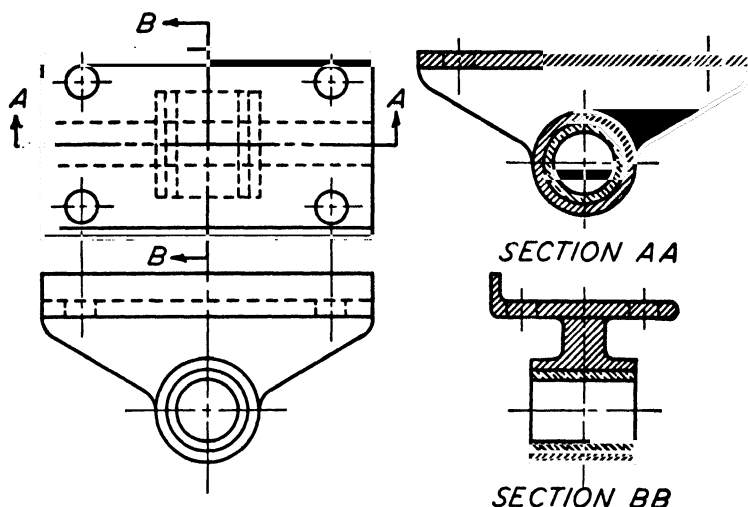


Fig. 30. Cast-Iron Spring Bracket

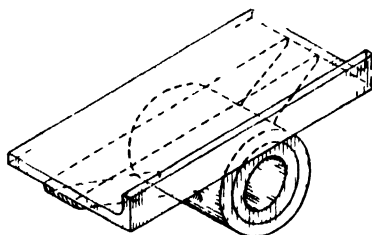


Fig. 31 Sketch of a Cast-Iron Spring Bracket

Sections are not always taken on straight lines. Sometimes the section is taken on a zigzag line. The draftsman should always make his drawing so that it will indicate as plainly as possible the construction of the object.

BLUEPRINT READING

Fig. 32 shows the convention for materials, such as cast-iron, wrought iron, etc. Some drawings use one convention only for

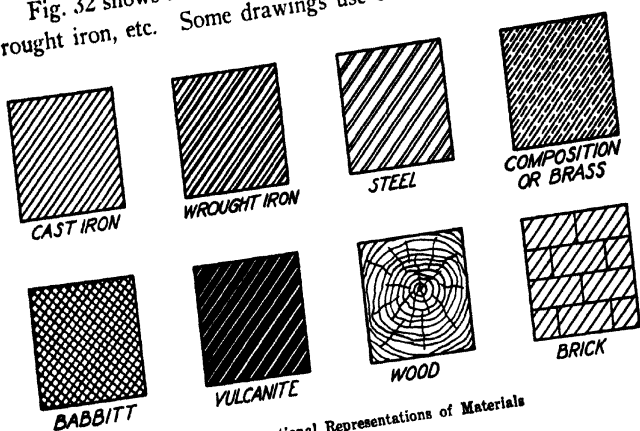


Fig. 32. Conventional Representations of Materials

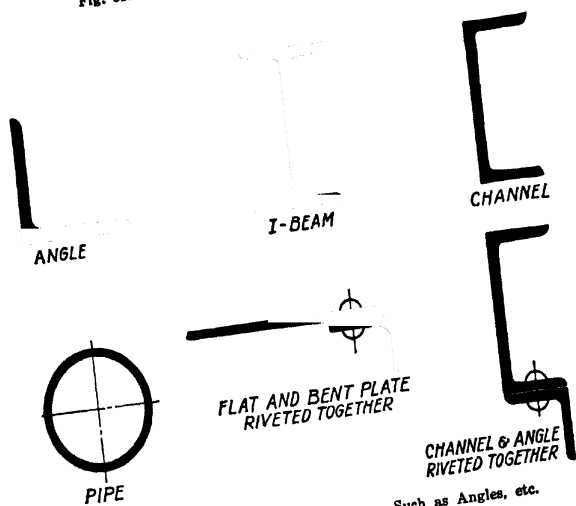


Fig. 33. Cross Sections of Thin Sections, Such as Angles, etc.

all materials as shown for cast-iron in Fig. 32; then the material required is given in notes on the drawing. Fig. 33 shows cross sections for thin materials, such as structural shapes, sheet steel, etc.

Where the object is of uniform shape, its full length is not always shown, then the view has a break in it. Fig. 34 illustrates the conventional methods of showing breaks.

To illustrate the interior construction of the object, cross-sectional views will be used.

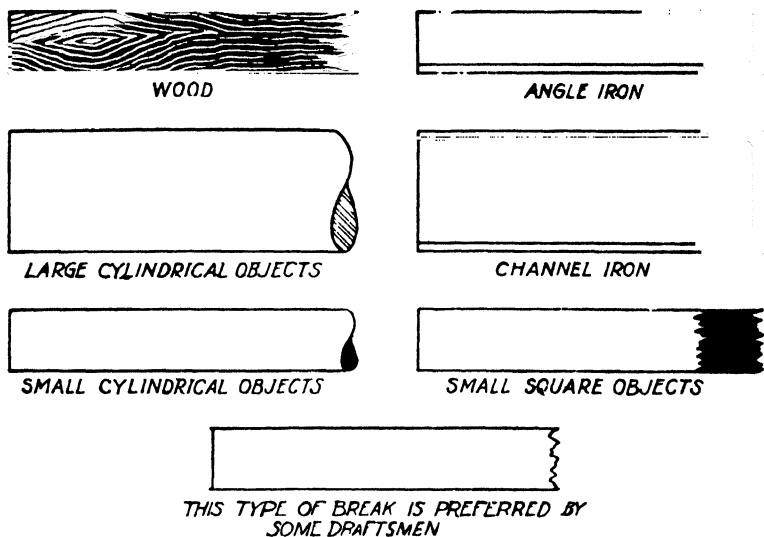


Fig. 34. Conventional Methods of Showing Breaks

Bell Ringing Transformer Case. The three views in Fig. 35 consist of a front elevation (view), a plan (top) view, and a right-side cross-sectional elevation (view).

It will be seen on referring to the cross-sectional elevation that the section is shown in a different manner than usual. The reason for this is that the material to be used for this piece part is comparatively thin, approximately $\frac{3}{64}$ ". Therefore, the cross section is shown as a solid black, where as in cross sections where the thickness is approximately $\frac{3}{32}$ ", etc., the regular cross-sectional convention lines would be used.

Let us see just how the die maker would use the drawing shown in Fig. 35. The piece part is made by a forming process with a punch and die on a punch press. But previous to this process, the punch and die (tools) have to be made. He must

You will find at the side of the plan the specification, "Punch .190"—2 holes," which is the size of the holes and also the size of the punch. On some drawings the distance between centers of the .190" holes might be specified on the front elevation and on others in the plan view. In this particular case, you will find the center distance on the plan view. It is to be $1\frac{1}{8}$ ". Also you will find on the plan view the dimension $\frac{3}{8}$ ", which is the distance from the end of the case one of the holes is to be located. To the left of the perpendicular center line, you will find the radius of the convex part is to be $\frac{9}{16}$ ".

In making the punch and die, it is very necessary that the mechanic know the thickness of the stock that is to be used for the piece part that is to be punched and formed.

On some drawings you might find this specified on the front elevation and on others on the cross-sectional elevation. In this particular case, you will find the thickness of the stock on the cross-sectional elevation. It is to be No. 19—.0438 United States Standard Gage. The 19 is a gage number. In some shops the thickness of the stock is referred to by certain numbers; then in other shops the number of the stock which is the gage is specified; and in others the gage number and the equivalent thickness of stock will be specified. The .0438" is the equivalent thickness of No. 19 United States Standard Gage stock.

QUIZ QUESTIONS*

1. What should be the diameter of the punches for the two large holes?
2. What gage and thickness of stock should be used for the bell ringing transformer case?
3. What is to be the length of the transformer case?
4. What is the distance from the side of the transformer case to the center of the $\frac{1}{2}$ " holes.

*No written answers are required for Quiz Questions.

Automotive Front Spring Bracket. Fig. 36 is a working drawing of an automobile front spring bracket. Three views are shown—a front elevation, a plan view, and a cross-sectional

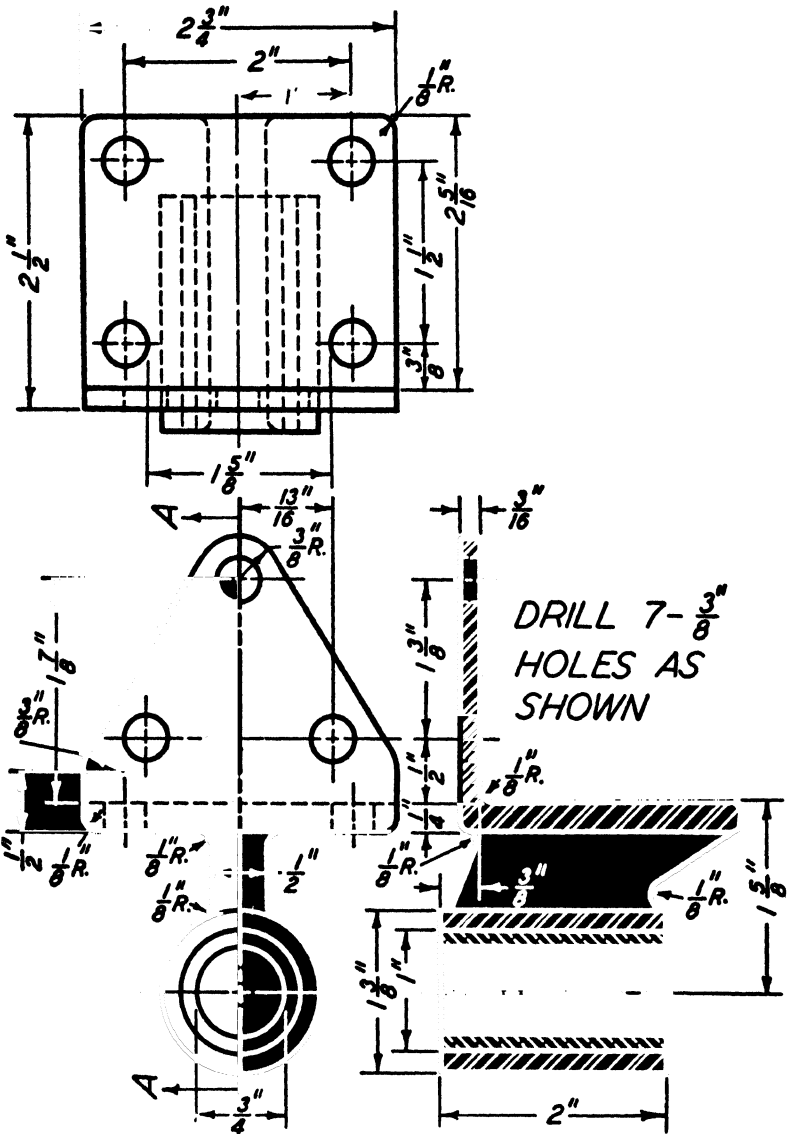


Fig. 36. Automotive Front Spring Bracket

elevation. You will notice that this cross-sectional elevation is of a different type than that shown for the transformer case. You will also notice that part of Fig. 36 has been cross sectioned the same as the convention used for steel, as shown in Fig. 32; and that the piece inserted in the hub is cross sectioned the same as the convention for composition or brass, as shown in Fig. 32. This same convention is also used on a bushing, indicating phosphor bronze.

Now let us consider Fig. 36 from the pattern maker's viewpoint. He will want to know the distance from the center of the $1\frac{3}{8}$ " hub to the flat part of the bracket or that part that is $2\frac{5}{16}$ " x $2\frac{3}{4}$ "; the length of the hub; the thickness of that part of the casting that connects the hub to the upper part; and the size of the radii.

Referring to the cross-sectional elevation, you will find that the distance from the center of the hub to the flat part is to be $1\frac{5}{8}$ ". Also you will find directly below the hub that the length of the hub is to be 2". In order to find the thickness of that part of the casting that connects the hub to the upper part, refer to the front elevation and you will find that this thickness is to be $\frac{1}{2}$ ". You will also find on the front elevation that the radius for the upper part of the bracket, that is, the section of metal outside of the upper hole, is $\frac{3}{8}$ ".

As it would not be good practice to have square corners between the upper part of the casting and the $\frac{1}{2}$ " rib, a fillet must be put in on each side of the $\frac{1}{2}$ " section, that is, at the top of the rib and at the bottom of the rib that connects the hub. Referring to the front elevation, you will find that these radii are to be $\frac{1}{8}$ ". Fig. 37 will help you to have a better understanding of the design of Fig. 36.

QUIZ QUESTIONS*

1. What is to be the length of the bronze bushing?
2. What is to be the diameter?
3. How many small holes are to be in the spring bracket?
4. What is to be the diameter?
5. What are the center distances of the four $\frac{3}{8}$ " holes?

*No written answers are required for Quiz Questions.

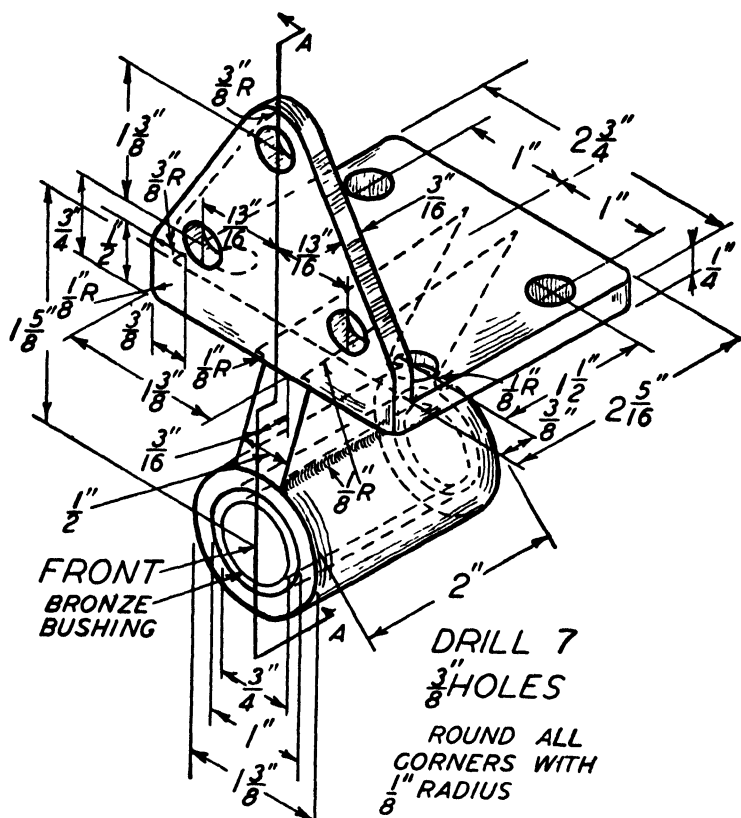


Fig. 37. Sketch of an Automotive Spring Bracket

Steering Column Bracket. The working drawing of the steering column bracket, Fig 38, has four views—a front elevation, a plan view, a right-side elevation, and a sectional elevation *AA*. Fig. 39 is a sketch of this bracket which will enable the student to have a better understanding of Fig. 38.

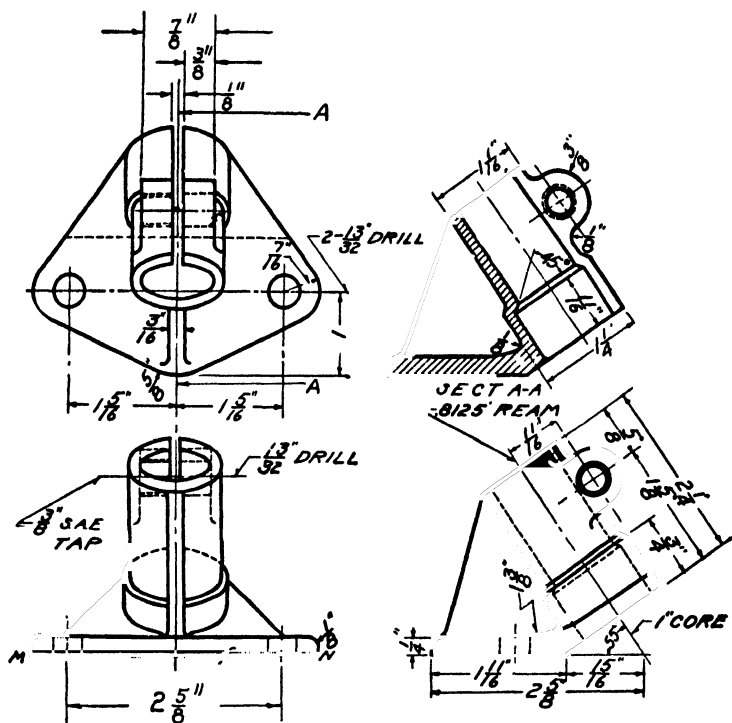


Fig. 38. Steering Column Bracket

You will note on referring to Fig. 32, that the convention for cast iron is used in the sectional elevation in this working drawing.

Machining the steering column bracket would be considered a manufacturing job as the pieces would be machined in quantities. Therefore, let us consider the equipment that would be used to produce such pieces efficiently and cheaply.

First, the drill jig must be made. This is part of the tool maker's job. Therefore, the tool maker would study the design of the piece part in order to get an idea of the outside construction. Then he would study Section *AA* to find out about the interior construction.

The first drill jig to be made is the one that is to be used for machining the hole that is in the hub. The tool maker must find out whether this hole is to be drilled or reamed and whether it is to be machined all the way through. Directly above the right-side elevation you will find the specification ".8125" ream," which means that this finished hole is to be

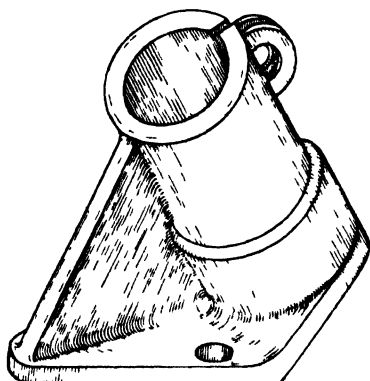


Fig. 39. Sketch of a Steering Column Bracket

an accurate one and must be machined out by a reaming process. Also on the right-side elevation you will find that this hole is only machined part of the way through owing to the fact that at the other end of the hub there is a cored hole part of the way through. In order to find out what the depth of this cored hole is to be, refer to the cross-sectional elevation which shows the construction of the interior of the piece part, and you will find the depth of the cored hole is to be $1\frac{1}{8}$ ". The cored hole is a rough hole that is made in the foundry or a hole that is not machined.

Let us assume that this jig has been completed. Now the tool maker is ready to start machining the other holes.

On referring to the plan view, he finds that there are two holes to be drilled out with a $\frac{1}{2}$ " drill. In preparing the jig, he must know the distance from the end of the bracket to the center of the two holes.

On the plan view, you will find that these holes are to be located one inch from the end.

Now the tool maker must find out what the distance is to be from the center of the .8125" reamed hole to the center of each of the $\frac{1}{2}$ " holes.

In referring to the plan view, you will find that these centers are to be $1\frac{5}{8}$ ".

As the tool maker plans to drill and tap the split section of the bracket in the same jig, it is necessary for him to know the size of the drill and tap that must be used.

On the front elevation, you will find that one side of the split section is to be drilled out and that the drill is to be $\frac{1}{2}$ "; that the other side is to be tapped or threaded and that the size of the tap is to be $\frac{3}{8}$ " S.A.E. tap.

NOTE: The $\frac{3}{8}$ " S.A.E. tap is a standard $\frac{3}{8}$ " tap, made according to the Society of Automotive Engineers Standard threads. This particular type of tap produces a comparatively fine thread, having a $\frac{1}{16}$ " pitch. The pitch is the distance from the center of one thread to the center of the next thread. There are also United States Standard threads. The pitch for a $\frac{3}{8}$ " United States Standard thread is $\frac{1}{8}$ ". In other words, there are 16 threads per inch.

Another thing the tool maker must know is the location of the holes that are to be tapped and drilled.

On referring to the right-side elevation, you will find that these holes are to be located $1\frac{1}{2}$ " from the bottom of the hub and $\frac{5}{8}$ " from the top of the hub and $\frac{1}{4}$ " from the center of the .8125" reamed hole.

The tool maker is now ready to use the milling fixture for machining the surface *MN*. He must know what the thickness of the bottom of the bracket is to be and at what angle the .8125" hole is to be to the bottom surface of *MN*, so that the milling fixture can be made in such a way as to maintain this angle.

Referring to the right-side elevation, you will find that the thickness of the bracket is to be $\frac{1}{4}$ ". Also you will find

at the bottom of this right-side elevation, that the angle the bottom surface of *MN* is to be held within is 55° .

QUIZ QUESTIONS*

1. What is the depth of the cored hole in the steering column bracket?

2. What is the distance from the lower end of the hub to the tapped hole?

3. What is the distance from the center of the reamed hole to the center of the tapped hole?

4. At what angle is the reamed hole from the base?

5. What size of reamer would you use for the reamed hole?

***No written answers are required for Quiz Questions.**

Worm Gearing. This worm gearing consists of a worm-wheel and a worm in mesh. The working drawing of the worm gearing, Fig. 40, has two views—a sectional elevation, indicated as "*SEC. AB*," and a side elevation. In the side elevation, the upper part of the worm indicating the threads and the lower part of the worm-wheel are shown in section in order to make clear the design of the teeth. Figs. 41 and 42 give the names of the various parts of the gear and worm.

Referring to the sectional elevation, you will find that a cutting plane has been taken on line *AB*, as shown on the side elevation. The cutting plane cuts the worm-wheel and worm into two pieces and is taken exactly through the center of the worm-wheel and worm. The circle, which is indicated by dotted lines, represents the bottom of the worm thread. The circle between the invisible and the outer circle is one with a dash and a single dot, which indicates that this is a center line or that it has been used for indicating the pitch diameter of the worm. The outside diameter of the worm is indicated by a full circle also the hole in the worm.

In Fig. 40, the outside diameter of the worm-wheel is indicated by a dash and double dot, but some draftsmen use a full line to indicate this circle. The throat diameter of the worm-wheel, which is 5.4112", is indicated by a dotted circle. The pitch diameter, which is 5.0929", is indicated by a dash and

dot circle. The bottom of the worm-wheel teeth is indicated by the same type of line as the outside diameter. In the worm gearing, Fig. 43, arms are shown; whereas in Fig. 40, $\frac{3}{8}$ " web is shown.

Now let us study this working drawing, Fig. 40, in connection with the job of the lathe hand. First, he must know the diameter to which the worm is to be turned and the length of the worm.

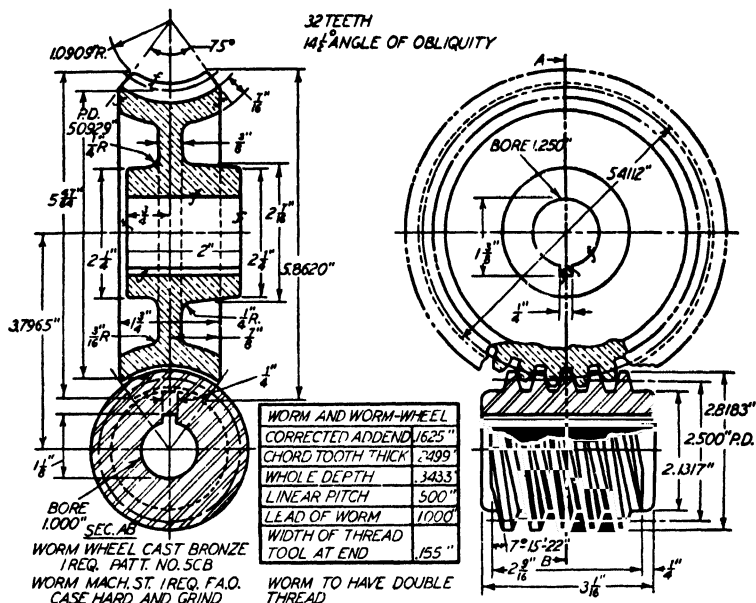


Fig. 40. Working Drawing of a Worm Gearing

Referring to the side view, you will find the specification 2.8183". This is the diameter to which the worm is to be turned (machined). You will also find that the length of this worm, which is to be finished all over, is to be $3\frac{1}{16}$ ".

Before chasing (cutting) the threads on the worm blank, the lathe hand must find out just how to grind his cutting tool.

Between the sectional and side elevations in the tabulation, you will find the width of the threading is to be .155", which means that the end of this tool should be ground to a

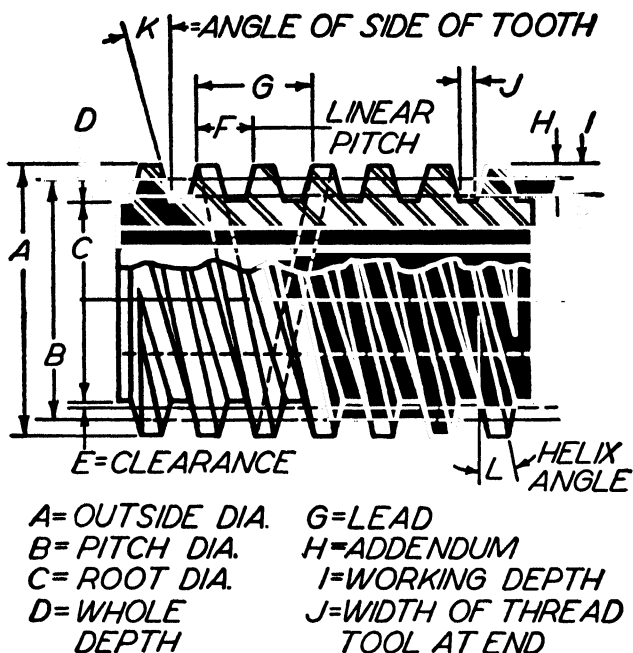


Fig. 41. Nomenclature of Worm

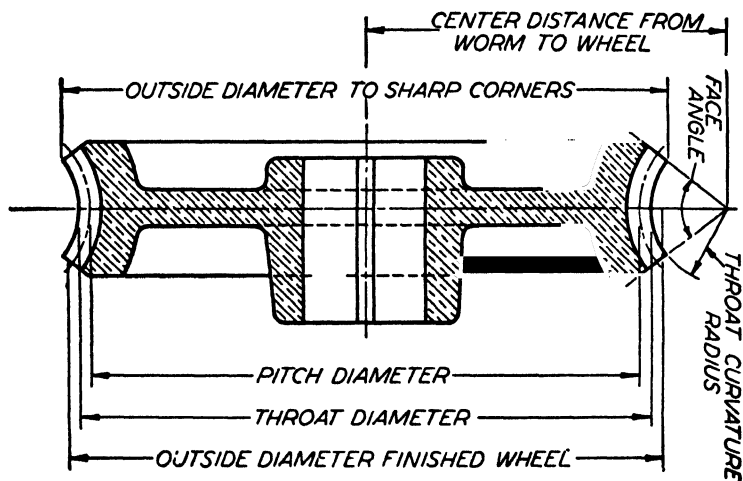


Fig. 42. Nomenclature of Worm-Wheel

width of approximately $\frac{5}{16}$ ". But to maintain the desired accuracy, the tool must be ground .001" less, or 0.155", the same as specified in the tabulation.

The lathe hand must now find out about the lead, and whether the thread is to be single or multiple.

You will find in the tabulation that the pitch is .500", and therefore, as the lead is twice the pitch, the lead is to be 1.000". The lead being twice the pitch indicates that the worm is multiple threaded or one having a double thread.

Let us now consider just what information the milling



Fig. 43. Worm Gearing

Courtesy of W. A. Jones Foundry & Machine Company

machine hand must have in order to machine the teeth on the worm-wheel. First, he must know how many teeth are to be cut on the blank; the pressure angle of the worm-wheel; and the pitch diameter of the worm-wheel.

At the top of the working drawing, you will find that there are to be 32 teeth and that the angle of obliquity is $14\frac{1}{2}^\circ$. In order to find the pitch diameter of this worm-wheel, refer to the sectional elevation and you will find the specification P.D. 5.0929".

Now, let us consider what information the lathe hand must have to machine the blank for the worm-wheel. First, he

must know the outside diameter, that is, to *sharp* corners of the worm-wheel; and the trimmed diameter.

Referring to the sectional elevation, you will find at the right the specification 5.8620", which is the dimension for the outside diameter. You will find on the left side of the sectional elevation the specification $5\frac{1}{4}"$, which is the dimension for the trimmed diameter.

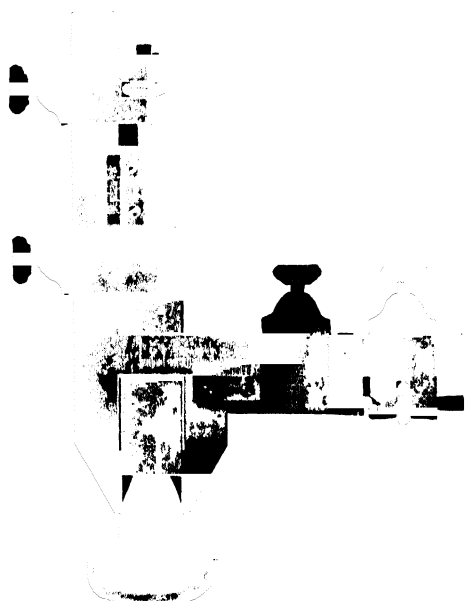


Fig. 44. Measuring Chordal Tooth Thickness and Corrected Addendum
Courtesy of Brown & Sharpe Manufacturing Company

In facing the ends of the hub, the lathe hand must know the length of the hub; also the diameter of bore of the hub; and the width of the worm-wheel.

Directly below the horizontal center line through the hub in the sectional elevation, you will find that the hub is to be 2" in length. Now, in order to find the diameter or bore (machine) of the hub, refer to the side elevation and directly above the hub you will find the bore is to be 1.250". In order to find the width of the worm-wheel, refer to the sectional elevation and you will find directly below the hub that this width is to be $1\frac{1}{4}"$.

It is extremely important to know the corrected addendum and chordal tooth thickness, when machining the teeth. The corrected addendum and chordal tooth thickness are measured with an instrument, as shown in Fig 44, and you will find the dimensions in the tabulation between the two views, Fig. 40. The dimension for the corrected addendum is to be .1625" and the chordal tooth thickness is to be .2499".

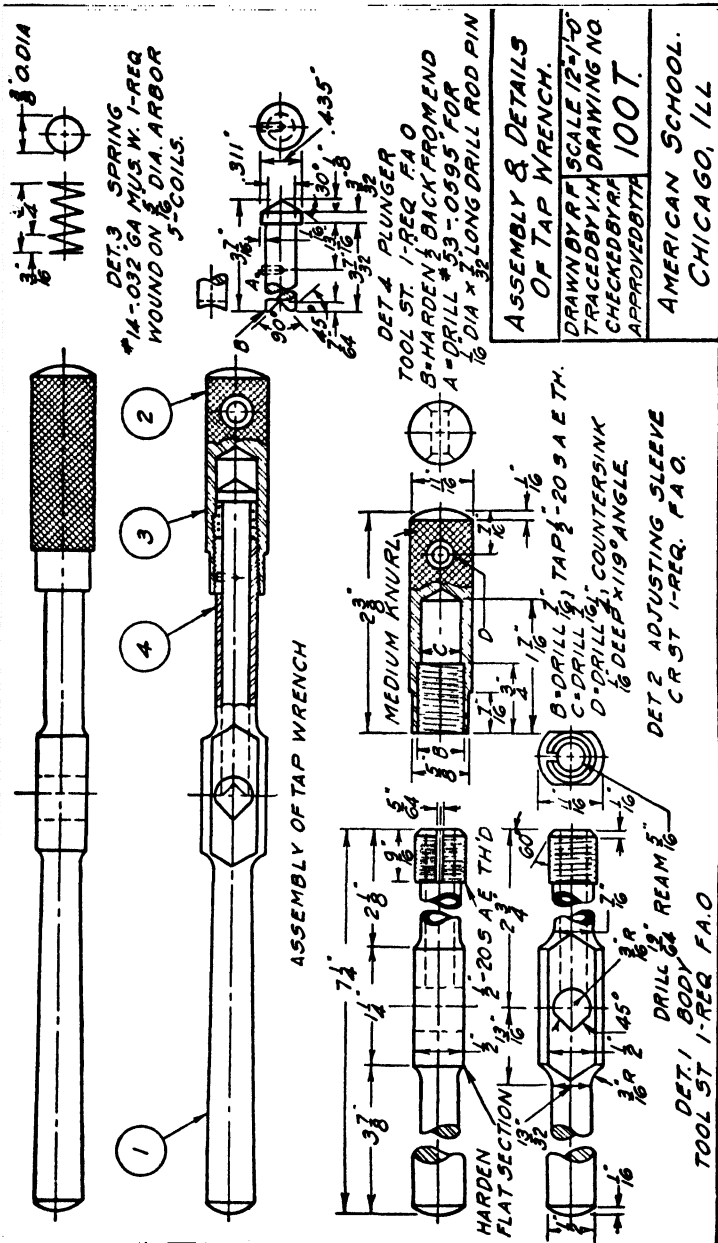
The mechanic who cuts the keyway in the worm-wheel may be a milling machine operator, a shaper hand, or a key-seating machine operator. First, he will need to know the width of the keyway and the depth.

Refer to the side elevation and you will find that the width of the keyway is to be $\frac{1}{4}$ ". The distance from the near side of the hole to the bottom of the keyway, as is specified on some working drawings, is not specified on this drawing. The specification given on this drawing is the distance from the far side of the hole to the bottom of the keyway, which is a more accurate way of measuring and gives more satisfactory results when an accurate depth of keyway must be provided for. You will find that the distance from the side of the hole to the bottom of the keyway is to be $1\frac{1}{8}$ ".

QUIZ QUESTIONS*

1. How many teeth are to be cut on the worm-wheel blank?
2. What type of line indicates the throat diameter of the worm-wheel?
3. What is the pitch diameter of the worm-wheel?
4. To what diameter would you turn the worm-wheel blank?
5. To what length would you machine the worm-wheel hub?
6. To what diameter would you bore the worm-wheel hub?
7. To what dimensions would you set the measuring instrument, Fig. 44, for measuring the chordal tooth thickness and the corrected addendum?
8. To what width would you grind the end of your tool for cutting the worm thread?

*No written answers are required for Quiz Questions.



ASSEMBLY DRAWINGS

Tap Wrench. Fig. 45 shows a typical assembly drawing and details of a tap wrench. The upper view in the upper left-hand corner of the plate is the plan view and the lower view is a front elevation of the assembly of the tap wrench. Part of the right end of the front elevation shows a section of the body, spring, and adjusting sleeve.

In the assembly drawing, the sectional view of the body indicates that it is to be made of steel and that the adjusting sleeve is to be made of steel. On examining the front elevation, especially that part of it that is to the right of the perpendicular center line, you will find that the plunger fits in the end of the body and that between the right end of the plunger and the right end of the body is a spring that is to be made from No. 14—.032" gage music wire. The inside of the adjusting sleeve being threaded affords an opportunity for turning or adjusting the sleeve. The small pin in the plunger is the same as shown in the "Det. 4" drawing at *A*. This pin slides in a slot in the right end of the body of the tap wrench and is used to prevent the plunger from turning.

In the specifications below the working drawing for the body, you will find "Det. 1"; below the adjusting sleeve, "Det. 2"; below the spring, "Det. 3"; and below the plunger, "Det. 4".

As the assembling of these pieces is a comparatively simple job, owing to the fact that there are not many piece parts required, it is possible to assemble the tap wrench without any reference numbers. But if there were many piece parts involved in the job, it would be absolutely necessary that the same reference numbers appear on the assembly drawing as specified for the details.

Reference numbers are used on this particular job in order to give the student an opportunity to get a better understanding of how such reference numbers are used in connection with an assembly drawing or in assembling a job in the shop.

In studying the details involved in the working drawings of this particular job, let us consider the body of the tap wrench first. Referring to the plan view of the working draw-

ing, you will find that the center part of the body, which is $1\frac{1}{4}" \times \frac{1}{2}"$, is to be hardened; and that the body is to be of tool steel and is to be machined all over. You will find the size of the hole in which the plunger is to fit and whether it is to be just drilled or whether it is to be drilled and reamed, directly below the front elevation. The specification says: Drill $\frac{3}{8}"$, ream $\frac{5}{16}"$.

On referring to the front elevation of the adjusting sleeve, you will find that the dimensions are not specified for the tapped and drilled holes, but the tapped holes are specified by the letter *B* and the drilled hole by the letter *C*; also you will find that the hole at the end of the adjusting sleeve is specified by the letter *D*. Examining the specifications below the drawing, you will find that *B* indicates that the hole is to be first drilled out with a $\frac{7}{16}"$ drill, subsequently tapped with a $\frac{1}{2}"$ —20 S.A.E. TH tap, meaning the hole is to be threaded with a $\frac{1}{2}"$ —20 Society Automotive Engineers Standard tap. You will also find that a $\frac{7}{16}"$ drill is to be used for hole *C*, and a $\frac{1}{4}"$ drill and a $\frac{7}{16}" \times 119^\circ$ angle countersinking tool is to be used for hole *D*.

If the spring, "Det. 3," is to be wound on a lathe, you will find what size of arbor, or mandrel as some mechanics call it, is to be used from the specification below the drawing, also the number of coils that must be made for this spring.

In setting up the job for winding the spring, you will find, on referring to the front elevation of the spring, that the pitch is to be $\frac{3}{16}"$ and the free height $\frac{3}{4}"$.

On the front elevation of the plunger, you will find at the left of the small pin that is inserted in the shank of the plunger the letter *A*, but the size of the pin is not specified. Referring to *A* in the specification, you will find that before the pin can be assembled, it will be necessary to drill a hole with No. 53—.0595" which would be suitable for a $\frac{1}{16}"$ diameter by $\frac{7}{8}"$ long drill rod pin. The number 53 is used for a certain size of drill. It frequently happens that on working drawings the draftsman only specifies the number of the drill. In this particular case, both the number of the drill and the diameter

of the drill are specified. The $\frac{1}{8}$ " diameter $\times \frac{7}{32}$ " long drill rod pin means that this pin shown at *A* in the front elevation is to be made from a piece of standard stock which is called drill rod, this being high grade round tool steel.

QUIZ QUESTIONS*

1. Which mechanic uses the reference numbers on an assembly drawing?
2. Must the detail numbers on an assembly drawing be the same as those used on working drawings?
3. What size drill and tap would you use for hole *B* in the adjusting sleeve?
4. What size arbor would you use for winding the spring (Det. 3)?
5. How much of the plunger (Det. 4) would you harden?
6. What kind of material would the body of the tap wrench be made from?

*No written answers are required for Quiz Questions.

Saddle Adjusting Lever. An assembly drawing of a saddle adjusting lever which has been used on a boring mill is shown in Fig. 46. This assembly drawing consists of a front elevation and a sectional elevation taken on the center line of the front elevation. In other words, the front elevation is theoretically cut in half and the right half removed, thus showing on the sectional elevation the interior construction of the lever.

The piece part that passes through spring 02268 and part 14387 is not shown in section on the sectional elevation but is indicated in the conventional way.

The shaft at the lower end of the lever is cross sectioned on the ends only, indicating that the actual length is longer. Shafts, pins, etc., are very seldom shown in section when used as they are in Fig. 46.

The numbers shown on the assembly drawing are piece part numbers. The use of such numbers on production work not only simplifies the clerical work in manufacturing plants, but it also simplifies the mechanic's job when putting the pieces together.

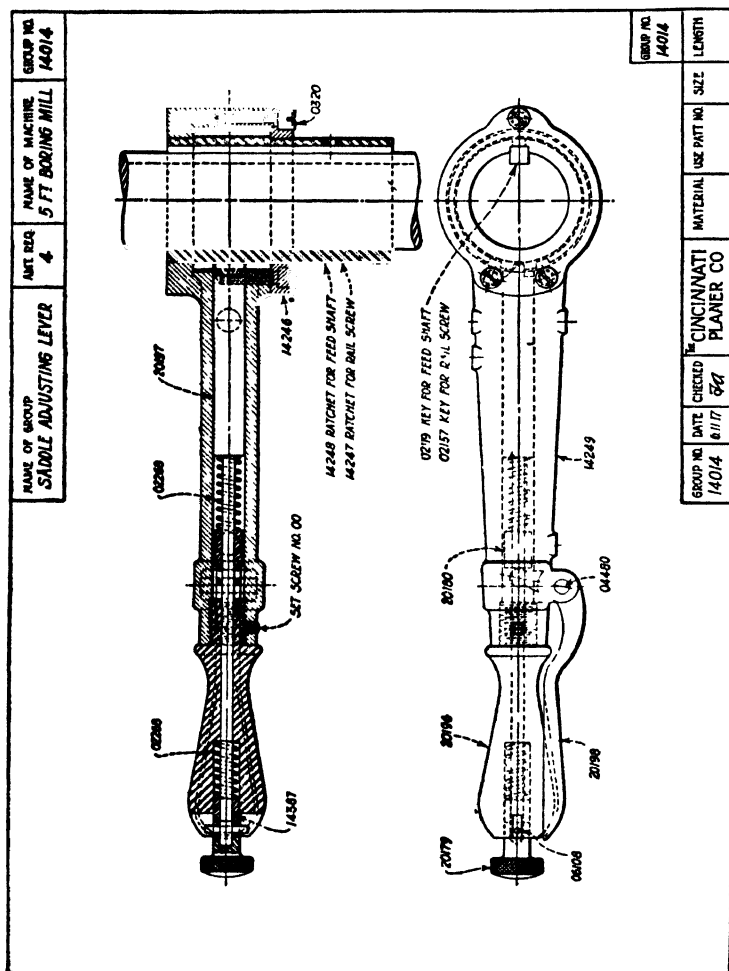


Fig. 46. Assembly Drawing of a Saddle Adjusting Lever

The conventions on the sectional elevation show that part 20196 is to be made of steel; that part 14249 is to be made of cast-iron; and that part 14248 and part 14247 are to be made of steel.

QUIZ QUESTIONS*

1. What are the numbers considered in Fig. 46?
2. Of what material is piece part 14387 to be made?
3. What is piece part 02268?
4. What does the crosshatching on 20179 indicate?
5. On what type of machine is the saddle adjusting lever used?

***No written answers are required for Quiz Questions.**

SCALE DRAWINGS

When and wherever it is possible and convenient to make the drawing of an object full or actual size, it should be so drawn. In many cases, however, the machine, or the building, or whatever is to be drawn, is so large that it would be impossible to draw it full size. Then the drawing is made to some reduced scale—that is, all the distances are drawn smaller than the actual dimensions of the object itself; all dimensions being reduced in the same proportion. For example, if a piece is to be drawn one-half size, the distance from one point to another on the drawing would be one-half what it is on the piece itself; if the drawing is one-quarter size, the distance on the drawing would be one-quarter what it is on the piece itself, and so on. In dimensioning such a drawing, the dimension written on the drawing is the *actual dimension* of the piece, and not the distance which is measured on the drawing. This fact must be very clearly understood by the student.

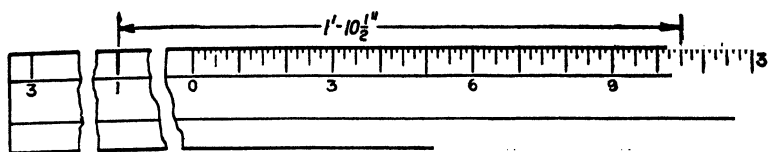
Methods of Reducing Dimensions. The common method of reducing all the dimensions in the same proportion is to choose a certain distance and let that distance represent one foot; this distance is then divided into twelve parts and each one of these parts represents an inch; then if half and quarter inches are required these twelfths are subdivided into halves,

quarters, etc., until the subdivisions become so small that they cannot be used. We now have a scale which represents the common foot rule with its subdivisions into inches and fractions; but our new foot is smaller than the ordinary distance which we call a foot, and of course its subdivisions are proportionately smaller. When we lay off a measurement on the drawing, we make it with our reduced foot rule; when we take a measurement on the machine itself, we make it with the standard foot rule.

Draftsmen's Scales and Their Use. Draftsmen's scales can be bought which have different distances thus divided, so that if the draftsman wishes to draw a piece one-quarter size, he looks over his scale until he finds a distance of 3 inches (which is one-quarter of a foot) divided as explained above, and he uses this to measure with on his drawing. His drawing would then be made to a scale of 3 inches to the foot. In the same way, if he wishes to make his drawing one-twelfth size, he finds on his scale 1 inch divided into twelfths and fractions of twelfths, and uses this as his standard of measurement; if he wishes to make his drawing one-forty-eighth size, he uses a quarter inch with its subdivisions.

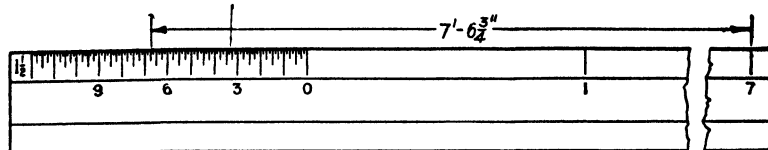
Fig. 47 demonstrates how the various scales are used. For instance, *A* shows the part of the scale where 3 inches equals 1 foot. To the right of the line marked *O*, in illustration *A*, is given the scale in inches and fractions of an inch. You will notice that subdividing this foot into twelve divisions you obtain inches; and subdividing the inches into eight divisions you will obtain a scale as small as $\frac{1}{8}$ inch. This illustration shows a reduction of 1 foot 10 $\frac{1}{2}$ inches (1'-10 $\frac{1}{2}$ ").

B indicates the part of the scale where 1 $\frac{1}{2}$ inches equals 1 foot. To the left of the line marked *O* is given the scale in inches and fractions of an inch. Subdividing this foot into twelve divisions you will obtain inches; and subdividing the inches into eight divisions you will obtain a scale as small as $\frac{1}{8}$ inch. This illustration denotes a reduction of 7 feet 6 $\frac{3}{4}$ inches (7'-6 $\frac{3}{4}$ "). Notice that the measurements start at the seventh division to the right of the *O* line, and six divisions to the left of the 6-inch line.



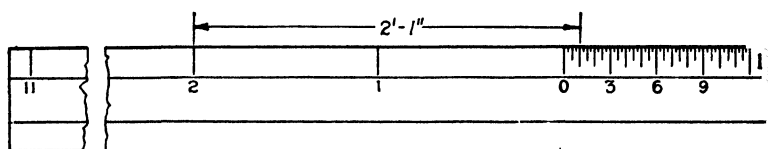
SCALE 3-INCHES = 1-FOOT

A



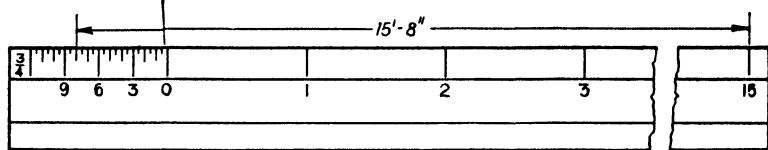
SCALE $1\frac{1}{2}$ -INCHES = 1-FOOT

B



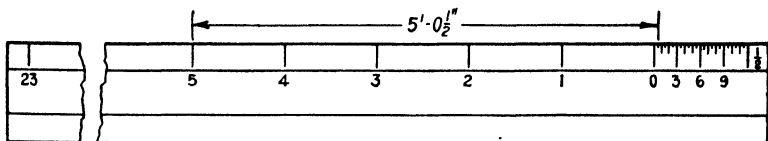
SCALE 1-INCH = 1-FOOT

C



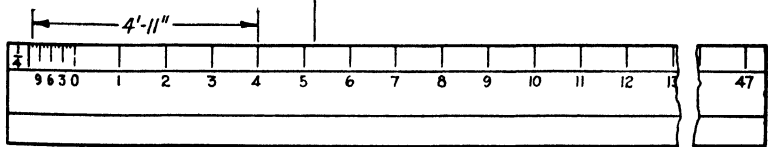
SCALE $\frac{3}{4}$ -INCH = 1-FOOT

D



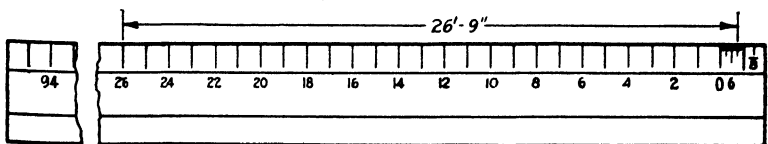
SCALE $\frac{1}{2}$ -INCH = 1-FOOT

E



SCALE $\frac{1}{4}$ -INCH = 1-FOOT

F



SCALE $\frac{1}{8}$ -INCH = 1-FOOT

G

Fig. 47. The Use of Various Scales

C shows the part of the scale where 1 inch equals 1 foot. To the right of the line marked *O* is given the scale in inches and fractions of an inch. Subdividing this foot into twelve divisions, you will obtain inches; and subdividing the inches into four divisions you will obtain a scale of $\frac{1}{4}$ inch. This illustration denotes a reduction of 2 feet 1 inch (2'-1"). Notice that the measurements start at the second division to the left of the *O* line, and four divisions to the right of the *O* line.

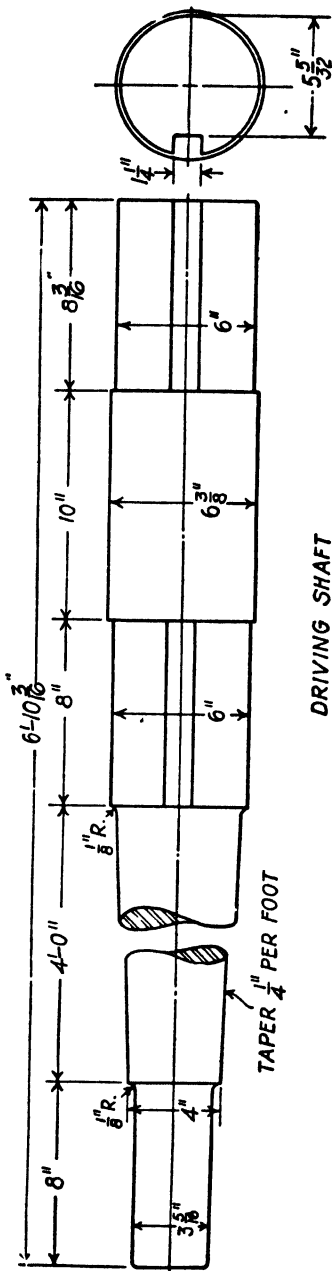
D shows the part of the scale where $\frac{3}{4}$ inch equals 1 foot. To the left of the line marked *O* is given the scale in inches and fractions of an inch. The smallest fractional division on this scale is $\frac{1}{2}$ inch. This illustration represents a reduction of 15 feet 8 inches (15'-8"). Notice that the measurements start at the fifteenth division to the right of the *O* line, and sixteen divisions to the left of the *O* line.

E shows the part of the scale where $\frac{1}{2}$ inch equals 1 foot. To the right of the line marked *O* is given the scale in inches and fractions of an inch. The smallest fractional division on this scale is $\frac{1}{2}$ inch. Here a reduction of 5 feet $\frac{1}{2}$ inch (5'-0 $\frac{1}{2}$ ") is indicated. The measurements start at the fifth division to the left of the *O* line, and one division to the right of the *O* line.

F shows the part of the scale where $\frac{1}{4}$ inch equals 1 foot. To the left of the line marked *O* is given the scale in inches only. A reduction of 4 feet 11 inches (4'-11") is here indicated. The measurements start at the fourth division to the right of the *O* line, and eleven divisions to the left of the *O* line.

G shows the part of the scale where $\frac{1}{8}$ inch equals 1 foot. To the right of the line marked *O* is given the scale in inches only. This shows a reduction of 26 feet 9 inches (26'-9"). Notice that the measurements start at the twenty-sixth division to the left of the *O* line, and nine divisions to the right of the *O* line.

Fig. 48 shows a detail which has been reduced in making the cut so that its dimensions are on a scale of $1\frac{1}{2}$ inches per foot, or one-eighth size. By applying the scale to this cut and comparing the readings on the scale with the dimensions as given, the student will gain a clear understanding of a reduced scale drawing.



DRIVING SHAFT
MACH. STEEL. 1-REQ. F. A. Q.

Fig. 48. Reduced Scale Drawing. Actual Scale, $1\frac{1}{2}$ inches equals 1 foot

Fig. 49 shows a detail which has been reduced in making the cut so that its dimensions are on a scale of 3 inches per foot, or one-fourth size. The scale can be applied to this drawing in the same manner as was done in Fig. 48

Fig. 50 shows a detail which has been reduced so that its dimensions are on a scale of 6 inches per foot, or one-half size.

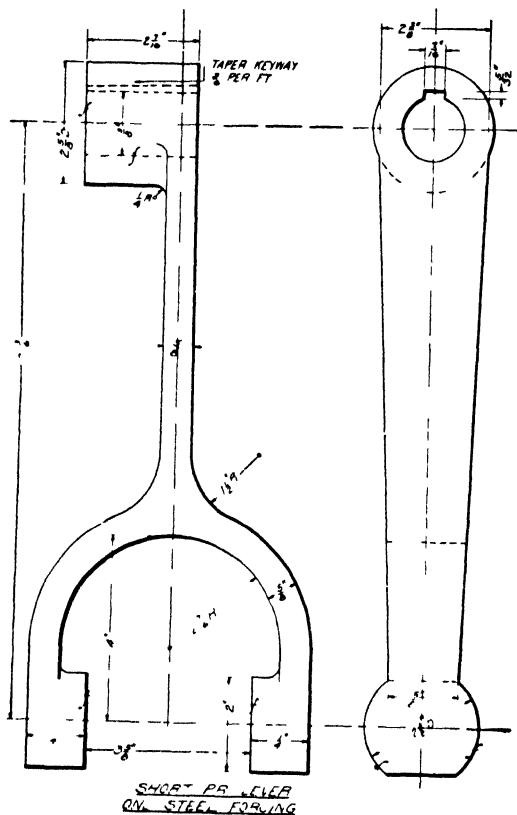


Fig. 49 Reduced Scale Drawing Actual Scale, 3 inches equals 1 foot

The scale can be applied to this drawing, as was done in Figs. 48 and 49.

The foregoing reduced details are merely illustrations for making comparisons with scale drawings. However, in practice, it is not a good policy to scale a blueprint, as the blueprint

shrinks after it is made. A drawing should always be made so complete that it is not necessary for the workman to use a scale.

Usual Scales for Drawing. The best scales for shop drawings in the United States are those readily derived from the common foot rule, such as full size : 6 inches = 1 foot ; 3 inches =

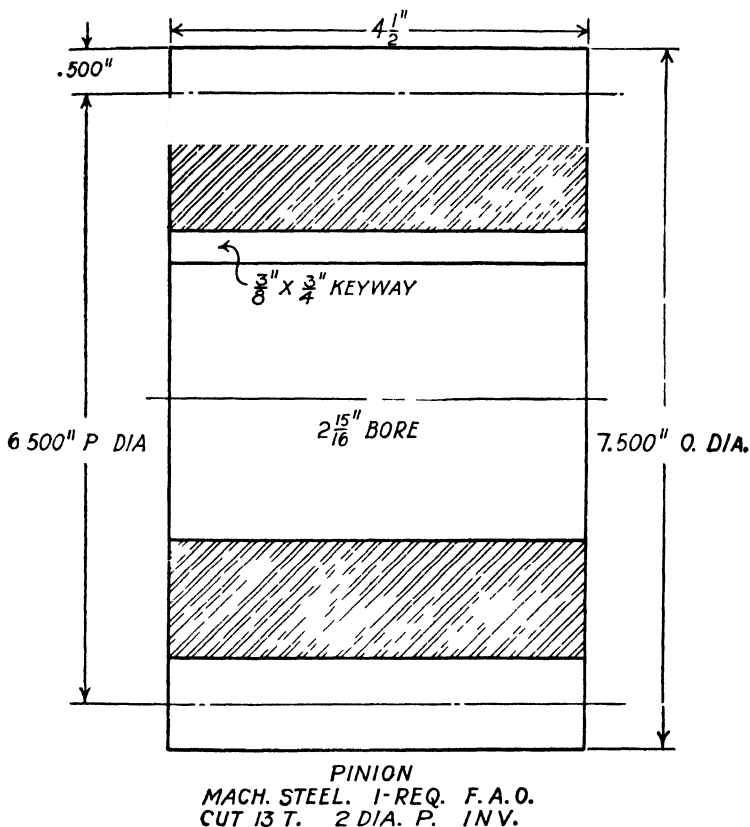
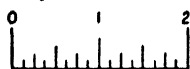
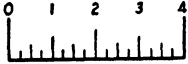


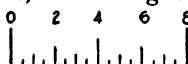
Fig 50 Reduced Scale Drawing. Actual Scale, 6 inches equals 1 foot

1 foot, $1\frac{1}{2}$ inches = 1 foot. These are the most common, most easily read from an ordinary scale, and one of these can usually be adopted. The student should learn to read these from an ordinary scale without being confined to a special graduation. In order to do this it is not necessary to divide each dimension by 2, 4, or 8 to get one-half size, one-fourth size, or one-eighth

size. For example, for one-half size, or 6 inches=1 foot, $\frac{1}{2}$ inch on an ordinary rule represents 1 inch. Hence, each $\frac{1}{2}$ inch may be read as 1 inch, and its subdivision accordingly, thus:



For 3 inches=1 foot, or one-fourth size, $\frac{1}{4}$ inch represents 1 inch, and looks thus:  for $1\frac{1}{2}$ inches

=1 foot, or one-eighth size, $\frac{1}{8}$ inch represents 1 inch, and looks thus:  It is very easy to get accustomed to this, and it saves much time and trouble hunting up a special scale every time.

The other allowable scales, less common but sometimes necessary on large work, are: 1 inch=1 foot; $\frac{3}{4}$ inch=1 foot; $\frac{1}{2}$ inch=1 foot; $\frac{3}{8}$ inch=1 foot; $\frac{1}{4}$ inch=1 foot; and $\frac{1}{8}$ =1 foot. For convenience in the use of these scales, special graduation is desirable.

Every drawing should have the scale to which it is made plainly marked upon it, as a part of, or adjacent to, the title.

Sometimes if the piece to be drawn is too small to be satisfactorily shown full size, the drawing is made to an enlarged scale, such as twice size, three times size, etc.

The mistake of choosing the wrong distance to use on a scale is often made. For example, if one wishes to draw a piece one-fourth size, he may look over his scale for a place marked $\frac{1}{4}$, and use this for his standard for one-fourth size, which is wrong. The figure on the scale indicates the distance which is divided up to represent 1 foot, so that the part of the scale which has $\frac{1}{4}$ marked on it means that $\frac{1}{4}$ of an inch is divided up into twelfths, or in other words, if a drawing is made according to that scale it will be one-forty-eighth size.

BLUEPRINT READING—Part II

—For the Machine Trades—

The discussions in this part of the text will be on the reproduction of actual shop drawings taken from some well-known manufacturing and machine building concerns in the United States. This has been done to make the study a practical one, so therefore, try to place yourself in the position of the various mechanics when using these drawings or blueprints.

WORKING AND ASSEMBLY DRAWINGS

Ball-Bearing Spindle. Fig. 51 is a comparatively simple blueprint of a ball-bearing spindle. Only one view is shown. The dimensions are indicated on or near it. This blueprint will be used in the machine building plant by a competent machinist.

From a drafting point of view, this blueprint could be criticized rather severely as it violates many standard conventions of good drafting; but it is shown because a mechanic occasionally has to use a blueprint of this kind.

A machinist before proceeding with his job must know the over-all length of the spindle. As a rule, the over-all length is given, but it is not specified on this blueprint so it must be determined. To the left of the cast-iron hub, which is on the right end of the drawing, he finds the dimension 1" and below it $1\frac{1}{8}$ ". Subtracting these dimensions, he gets $\frac{1}{8}$ ", which is the depth of the counterbore in the hub. Then he takes the distance from the left end of the spindle to the left end of the hub, which is $16\frac{1}{8}$ ", and adds to it the depth $\frac{1}{8}$ ", and the distance from the bottom of the counterbore to the end of the spindle, which is $2\frac{1}{8}$ ", and he gets $19\frac{1}{8}$ ", which is the over-all length.

Now the machinist must know the diameter and length of that part of the spindle which the No. 1209 SKF* bearing fits.

* SKF is the name of the ball bearings.

On examining the blueprint, he finds the diameter of the spindle is 1.7717" and the length is $\frac{3}{4}$ ". The hole in the end of the spindle is $\frac{5}{8}$ " and the size of the spindle for the SKF No. 707U ball bearing is 1.377". It will be noted that the size and location of the holes which are to be located in the assembly are not given. This means that these holes are drilled when the parts are assembled, so that accurate locations can be obtained. In all cases of uncertainty about dimensions, materials, etc., the safe procedure for the mechanic is to consult his foreman.

QUIZ QUESTIONS*

1. What is to be the diameter of the bore in the cast-iron hub?
2. Of what material is the spindle made?
3. What is the length of that part of the spindle which has $1\frac{1}{2}$ "-16 threads?
4. What is the length of that part of the spindle which carries $1\frac{3}{4}$ "-16 threads?
5. What is the size of the hole which is drilled in the end of the spindle?
6. What is the length of the part which fits into the No. 707U SKF ball bearings?

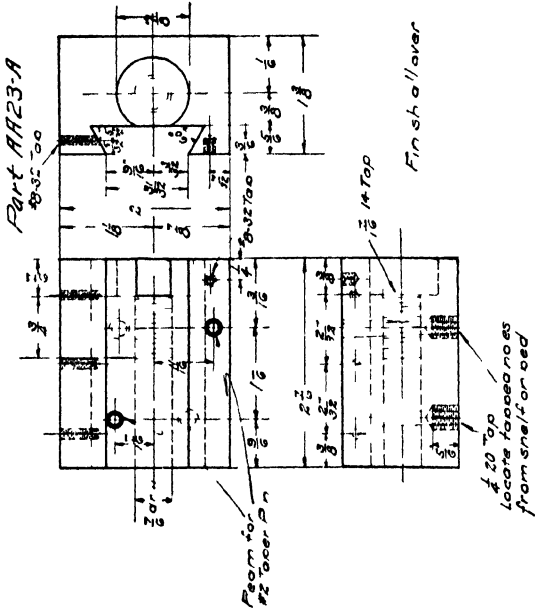
*No written answers are required for Quiz Questions.

Gauge Block. Fig. 52 is a blueprint of a gauge block for a halftone beveler or rabbeting machine. There are three views—the plan, front elevation, and right end elevation or, as often named, the top, front, and end views. As practices vary in different drafting offices, you may not find the end elevation always placed as it is shown in this blueprint. It could have been placed to the right of the front elevation.

The first thing the machinist will want to know before setting up his job in the shaper will be the thickness, width, and length of the gauge block.

In studying the blueprint, he finds that the thickness is to be $1\frac{3}{8}$ " and the width is to be 2". He finds these dimensions as part of the right end elevation. The length is $2\frac{7}{16}$ ". This dimension is shown between the front elevation and the plan or top view.

COMMERCIAL PARTS		MACHINE		LAT. M/T		ARTICLE		QUANTITY		SHEET	
# 20 1/2 HD CAP SCREWS		1 1/2" Long, 1/4" Dia. 4-40		TRAVER		3-102 BLOCK		1		AR23	
# 1 1/2 LG. WIPER PIN		1		STANDARD		WILLIAM R FIELD CO. CHICAGO					
# 2 1/2 LG. WIPER PIN		1		UNITS		OPERATION					
		C 1									
		MATERIAL									
		SECTION									
		ROUGH LENGTH									



In the plan view, there are two invisible horizontal lines which are the same length as the visible lines and, by referring to the right end elevation, the machinist finds that the two visible and two invisible lines indicate that the dovetail is to be the full length of the block, or $2\frac{7}{16}$ ".

The machinist must also know the depth of the dovetail and its angle. He finds the depth is $\frac{5}{16}$ " and the angle is 60° , as shown on the end elevation. He also notes that the dovetail is placed so that it is $\frac{9}{16}$ " behind the center line of the drawing and $\frac{3}{8}$ " in front of it, therefore, the distance between the edges or the width of the opening of the dovetail will be $\frac{1}{2}$ ".

In order to tap the holes which run into the dovetail, the machinist must know the depth and location of these holes. On examining the end elevation, he finds directly above it the specification No. 8-32 tap. On some blueprints, the size of drill is also specified. However, the machinists who are building this class of work generally do not need to have this specification as they know from experience the proper size of tap drill to use. If, however, you do not know, consult a table which gives this information or ask the foreman.

The dimensions for the location of the tapped holes are specified directly above the front elevation or below the over-all dimension of $2\frac{7}{16}$ ". The first hole shown in the front elevation is to be located $\frac{3}{8}$ " from the right end, and the second and third holes are to be $\frac{2}{3}$ " apart.

QUIZ QUESTIONS*

1. What is the length, width, and thickness of the block?
2. What is the spacing between centers of the three tapped holes that run into the dovetail?
3. What is the width at the narrow part of the dovetail?
4. What is the angle of the dovetail?
5. What is the size of the tapped hole that comes under or below the dovetail?
6. What is the size of the taper pins?
7. What is the diameter and depth of the counterbore below the dovetail?

*No written answers are required for Quiz Questions.

Bracket for Adjusting Screw. Fig. 53 is a blueprint of a bracket for an adjusting screw used in a 16 by 20 combination saw and trimmer. Four views are shown—plan or top view, front elevation, left end elevation, and sectional elevation taken on line AA, as indicated near the plan view.

The reading of this blueprint is not as difficult for the machinist as it is for the pattern maker, inasmuch as the machinist receives the casting and only has to compare it with the blueprint to get a good idea of the shape of the piece. The pattern maker, however, has nothing to go by except the blueprint, therefore, he must be able to visualize how the finished casting will appear. Let us see how he does this.

First, he carefully studies the plan view in conjunction with the front elevation and the end elevation to get an idea of the general shape of the piece. The plan or top view gives the outline of the shape of the bracket. It also shows there are various ribs under the main piece to add strength. The elevation shows the thickness of the top part and the depth of the rib.

One of the first important measurements is the distance from the back of the casting to the center of the hub in which the $\frac{3}{8}$ "-#11 tapped hole will be. This is shown on the plan as $7\frac{1}{8}$ ". The horizontal distance from the left end of the casting to the center of the hub is $2\frac{3}{4}$ ". The hub is $1\frac{3}{4}$ " in diameter and is $1\frac{1}{2}$ " deep.

The pattern maker must find out what the several invisible lines on the plan view indicate. The lines in the arm of the bracket show the width of the vertical rib which connects the hub and the back part of the casting. It is $\frac{5}{8}$ ". The invisible lines at the back of the plan view show there is another rib which has three pads. The pads are $\frac{1}{4}$ " on the face and the space between them is $2\frac{3}{8}$ ". The depth of these pads is shown in the front elevation and it varies from $1\frac{1}{4}$ " to $1\frac{3}{4}$ ". The thickness of the rib containing these pads is shown as $\frac{3}{8}$ ", and there is a $\frac{7}{16}$ " hole drilled through this rib between pads.

The pattern maker notices that a line crosses through the casting at AA. This is a cutting plane, so he looks for a section of that part of the casting. He finds on Section AA that the back part of the casting is to be $\frac{1}{2}$ " thick and that the casting is to be cored out to a width of $\frac{3}{8}$ ". The length of the back rib of the casting is shown in the plan view as $6\frac{1}{2}$ ".

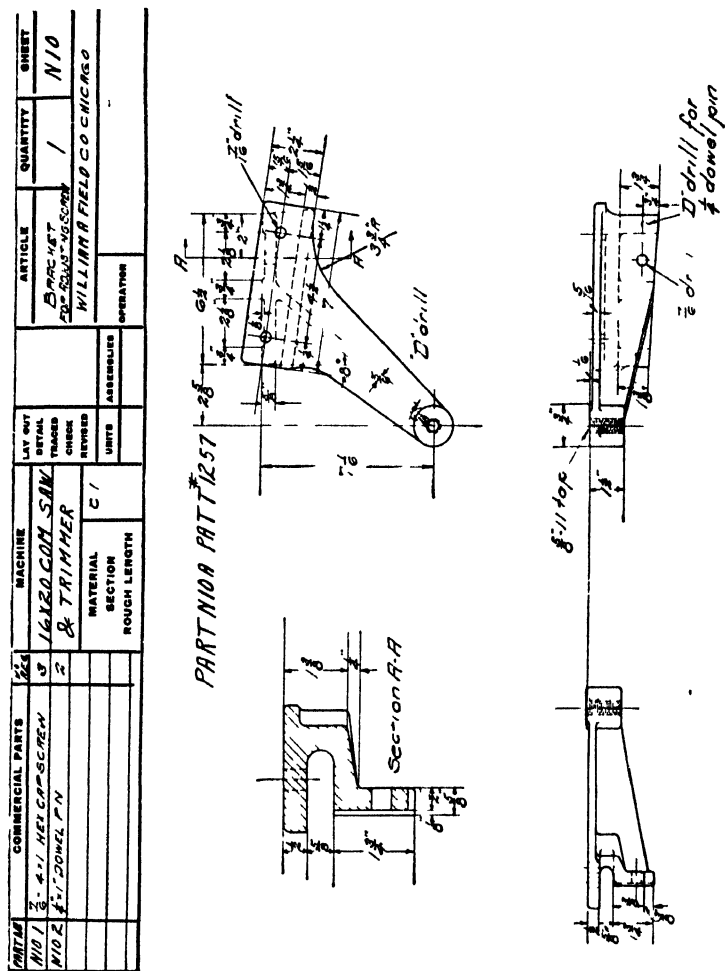


Fig. 53 Working Drawing for a Bracket for an Adjusting Screw

On examining the plan view again, the pattern maker notes there is another rib running across the bracket not quite parallel to the one containing the pads. This rib has a thickness of $\frac{1}{2}$ ". This plan view also shows the location of the two $\frac{7}{16}$ " holes which are drilled through the casting.

QUIZ QUESTIONS*

1. What is the size of the tapped hole in the hub at the end of the arm of the bracket?
2. What is the width of the rib that extends under the arm?
3. What is the thickness of the top part of the arm?
4. Of what material is the bracket made?
5. Give the location for the $\frac{7}{16}$ " drilled holes.
6. What is the thickness of the two ribs running across the casting?
7. What are the sizes of the pads on one of the ribs?

*No written answers are required for Quiz Questions.

Spindle Head. Fig. 54 is a blueprint for a wheel spindle head used on a tool grinding machine and Fig. 55 is an assembly drawing of the same machine. The spindle head job, in some cases, might be made in small quantities, but the blueprints should not be considered as being used strictly for machine building nor for manufacturing. The assembly drawing, Fig. 55, consists of two views—a front elevation and a right end elevation. In the front elevation, the upper part is shown in section, the central part is a full view, and the lower part is partly indicated. In the right end elevation, the central part is fully shown and the lower part is omitted. Practices vary in making assembly drawings. Some assembly drawings would have all the piece parts indicated by numbers, thereby aiding the mechanic to assemble the pieces; however, as will be seen from this blueprint, piece parts or detail numbers are not specified.

Let us consider blueprint Fig. 55 when it is being used in the assembly department as an aid to the mechanic in assembling the pieces. Before he proceeds to assemble the pieces, he makes a study of the blueprint. At the top of the front elevation of the machine, that is, the center part of it, he finds that some of the pieces are inclosed, and the piece that covers them

WHEEL SPINDLE HEAD

C.1. *N* 290

FIXTURE FOR MILLING TOP
FIXTURE FOR MILLING BOTTOM
JIG & TOOLS

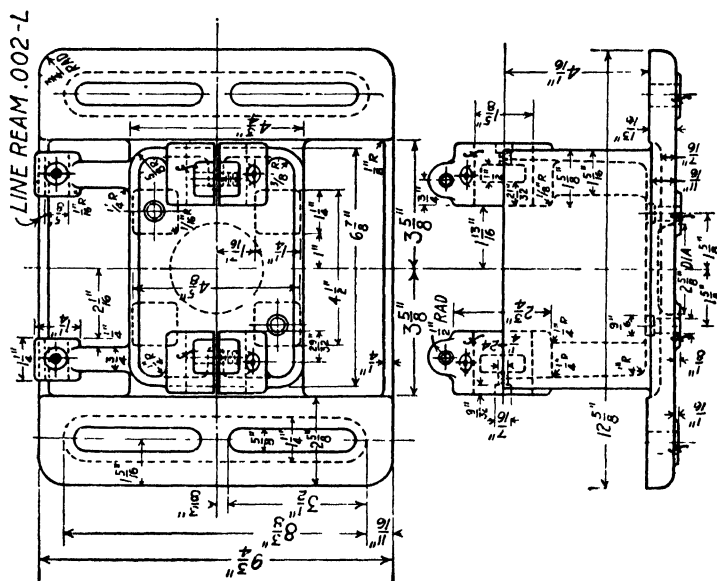


Fig. 54. Working Drawing for a Wheel Spindle Head
Courtesy of Brown & Sharpe Mfg Company

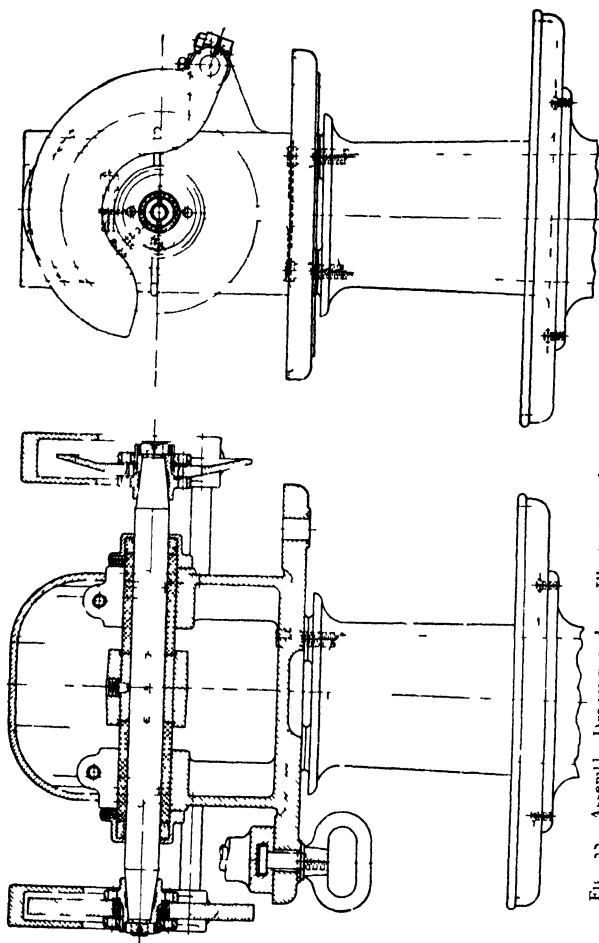


Fig. 22 Assembly Drawing of a Tool Grinder
Courtesy of Brown & Sharpe Mfg. Company

is the wheel spindle head. Within this wheel spindle head, which has bearing housings on each end with bushings inside of the housings, a spindle rotates. This spindle is to hold the emery wheels, as will be seen on the right and left ends of the spindle. Inside this wheel spindle head and between the bearings, a pulley is located, which, as will be seen from the section, is one that is made of cast iron and is held to the spindle by a headless set screw. To the right and left of the pulley are two pieces, shown in section, these being babbitted bushings which aid in keeping down friction. At the extreme left and right of the babbitted bushings, caps are shown, which come in contact with the right and left bearings. These caps are fastened to the bushings with headless set screws.

A better understanding of the spindle head may be had from a careful study of the right end elevation. It will be seen that the part of the head that is attached to the pedestal is grooved out somewhat at the top, this being indicated by invisible or dotted lines.

To the right and left of the center line of the machine there are lines tapering up from the bottom of the head to the bottom of the bearing, which indicate the sides of the ribs. The mechanic now refers to the sectional part of the front elevation and finds that the perpendicular lines directly below the bearings and setting back slightly from the inner ends of the bearings indicate the length of the ribs.

In the front sectional elevation, the mechanic finds that the upper part of the bearing housings are not crosshatched. So he refers to the right end elevation, at the upper part of the housings, and decides that the front sectional elevation was obtained by taking a cutting plane on the center line in the right end elevation. Here he finds two invisible perpendicular lines, one to the right and one to the left, extending from the top of the bearing down to the bushings. He also finds to the right and to the left of the top part of the housings a screw passing through it, which indicates that the housings are split at the top.

In the front elevation at the extreme right end of the spindle, the mechanic finds that a certain piece directly above and partly to the side of the wheel is shown in section. From the front sectional elevation, he is not able to determine just what this piece is; but in referring to the right end elevation,

he finds that the piece is a guard which is used to cover the top of the emery wheel. The purpose of the guard is to keep the flying pieces of a broken wheel from hitting the worker. How is this guard attached to the machine? In the right end elevation to the extreme right of the spindle head or back of it, the mechanic finds visible and invisible lines connecting to a circular part, which indicate that there is a bracket that connects the guard to the wheel spindle head. On the end of the arm is a boss with a hole through it in which a pin fits that is used for the guard to turn on when it is necessary to remove the wheel from the spindle. In other words, to remove the wheel from the spindle it is necessary to raise the guard out of the way.

The mechanic must now find how the wheel spindle head is fastened to the pedestal. In the front elevation on the right side of the pedestal and at the top of it, he finds a flat fillister head screw. In the right end elevation, he finds two such fillister head screws. This shows him that the wheel spindle head is fastened to the pedestal with four fillister head screws.

Let us now consider the blueprint from the point of view of a pattern maker. The pattern maker first studies his blueprint, Fig. 54, to find out what material is to be used and to get an understanding of the design of the wheel spindle head. His blueprint consists of three views—a plan view, a front elevation, and a right end elevation. On the right of the plan view, he finds directly below the specification wheel spindle head, C.I., which indicates that cast iron is the material and N 290 is the pattern number. The notes regarding "Fixture for milling top, etc.," refer to special tools which are available for mass production.

The pattern maker now looks for the width and length of the lower part of the spindle head. On the left of the plan view, he finds the width of the spindle head is $9\frac{1}{4}$ ", but he does not find the length given in the plan view of this blueprint. Therefore, he refers to the front elevation and finds directly above the lower part of the spindle head the dimension $12\frac{5}{8}$ ", which is the length.

In studying the front elevation, particularly the bottom part of it, the pattern maker finds a horizontal invisible line, which is located $\frac{7}{16}$ " from the bottom horizontal line. This shows him that the base has an impression in it, but it does

not give him the length and width of the impression, therefore, he makes the top part of the base $2\frac{1}{8}$ " on each end, and then allows the length of the impression to come as it will. From the study of the right end elevation, he finds the impression starts a short distance from the back and front side and terminates at the perpendicular line, showing the extreme width, or, in other words, the outside width of the box-like design on the casting, but the correct location cannot be obtained from the front elevation. Referring to the plan view, he finds that a $\frac{1}{4}$ " thickness is specified.

From the assembly drawing, Fig. 55, the mechanic finds that the housings are supported by ribs directly below them. To determine the distance from the outer edge of the vertical part of the casting to the edge of the rib, he refers to the front elevation, Fig. 54, between the right housing and the base and finds it is $1\frac{5}{16}$ ". He also finds near the front elevation directly to the side of the right housing that the distance from the center of the housing to the top of the base is $4\frac{1}{16}$ ". On some blueprints this dimension would be specified on the right side elevation and on others, on the front elevation.

The housings are located on the base and the pattern maker, by referring to the front elevation to the right of the perpendicular center line, finds that the dimension is $1\frac{1}{8}$ ". This is the distance from the center line to the inside face of the bearing housing. This dimension is used for locating the other housing. For the length of the housing he adds $1\frac{1}{8}$ " to $1\frac{1}{8}$ " which equals $3\frac{1}{8}$ ". He subtracts $3\frac{1}{8}$ " from $3\frac{1}{8}$ " which gives $\frac{3}{16}$ ". He then adds $\frac{3}{16}$ " to $1\frac{1}{8}$ " which gives $1\frac{3}{8}$ ". To the right of the left housing he finds the dimension $2\frac{1}{4}$ " which is the outside diameter of the housing.

The pattern maker may be accustomed to using drawings where the dimension for the hole in the housing would be shown on the right-hand elevation. In this particular case, he will find the dimension to the right of the front elevation near the right housing. The dimension of the diameter of the hole in the housing is $1\frac{1}{8}$ ".

After looking at the front elevation on the blueprint, the pattern maker is unable to judge whether or not part of the housings are enclosed in a box-like casting. He studies the right-hand elevation and learns from it that the bearings are partly encased in a rectangular box-like casting. He does not

know definitely just how this box should appear when looking at it from the top. So he finds it necessary to read the plan view. From it he finds that the corners of this box are rounded, so he must find the radius of the corners on the inside and outside. Below the right housing in the front right-hand corner of the box, he finds a $\frac{3}{8}$ " radius, which is the radius to use for the inner corner. The outside radius, as will be seen from the rear right-hand corner of the box, is $\frac{3}{8}$ ".

On the right end elevation directly above the housing, the pattern maker finds lugs that are to be machined through the center with a $\frac{3}{8}$ " saw, but he does not know their design or shape. He refers to the front elevation and studies the various parts of the bearing housing. He finds that the housing to the left shows that the ear or lug has a radius of $\frac{1}{2}$ ". Not knowing just how high the top part of this lug is to come, he again reads his blueprint but does not find the distance from the center of the housing to the center of the hole in the ear specified on the front elevation. Referring to the right end elevation, he finds this dimension $1\frac{9}{16}$ ", which gives the correct position to set his dividers when scribing the arc which will give the rounding for the ear.

The pattern maker now must find out how the emery wheel guard is supported. From the assembly drawing, Fig. 55, he learns that this guard swivels on a pin which is inserted in a hole in a bracket that is at the back of the box-like part of the casting. Now he must find the location of the bracket relative to the top of the box and the center of the box. As it is the center of the bracket in which the hole is machined, he looks for the location of the horizontal center line. On the right end elevation, Fig. 54, above the bracket and the box-like part of the casting is the dimension $4\frac{3}{8}$ ", which is the specification for the horizontal distance. Between the bearing and the right wall of the box, he finds the dimension $1\frac{1}{2}$ ", which is the distance from the top of the box to the center of the bracket in which the hole will be machined.

QUIZ QUESTIONS*

1. Of what material is the spindle bushings made?
2. What is the diameter of the bore through the bearing housing?

3. What is the diameter of the bore in the arms or brackets which carry the guard of the machine?
4. What is the distance between the inside faces of the main bearing housing?
5. How thick are the walls on the center box construction?
6. What is the height of the center box construction measured from the bottom surface of the spindle head?
7. What is the thickness of the bracket that is to carry the guard of the machine?

*No written answers are required for Quiz Questions.

Base for Machine Tool. Fig. 56 is a blueprint of a base for a machine tool. Four views are shown—a plan view, a front sectional elevation, a right end sectional elevation, and a partial right end elevation. From the blueprint it would be difficult for one to determine for what particular machine this base is to be used, but the illustration, Fig. 57, shows that this is a base for a shaper and that it supports the machine proper. The blueprint, Fig. 56, shows that the base is rectangular and that the casting is ribbed in various places.

We will first study the blueprint from the point of view of the pattern maker as it affords excellent practice of visualizing a job from a drawing. The pattern maker first must find the length, width, and height of the base. He studies the blueprint and finds at the right end of the plan view directly above the horizontal center line the dimension 28", which is the overall width of the base. Then just below the front sectional elevation, he finds the dimension 70½", which is the length of the base. At the right end of the front sectional elevation, he finds the dimension 10½", which is the extreme height of the base.

The right end of the drawing shows the low interior part of the base that connects to a solid mass of metal. This part of the base also connects to a perpendicular rib that is ¾" thick and continues on to a round corner. The round corner, in turn, meets another perpendicular rib. Now the pattern maker must find how far below the top of the base this part should be. On the right end of the elevation, he finds that this distance is to be 6", it being located to one side of the over-all height 10½". He also finds that the thickness of the part is to be ¼", this dimension being directly below the part.

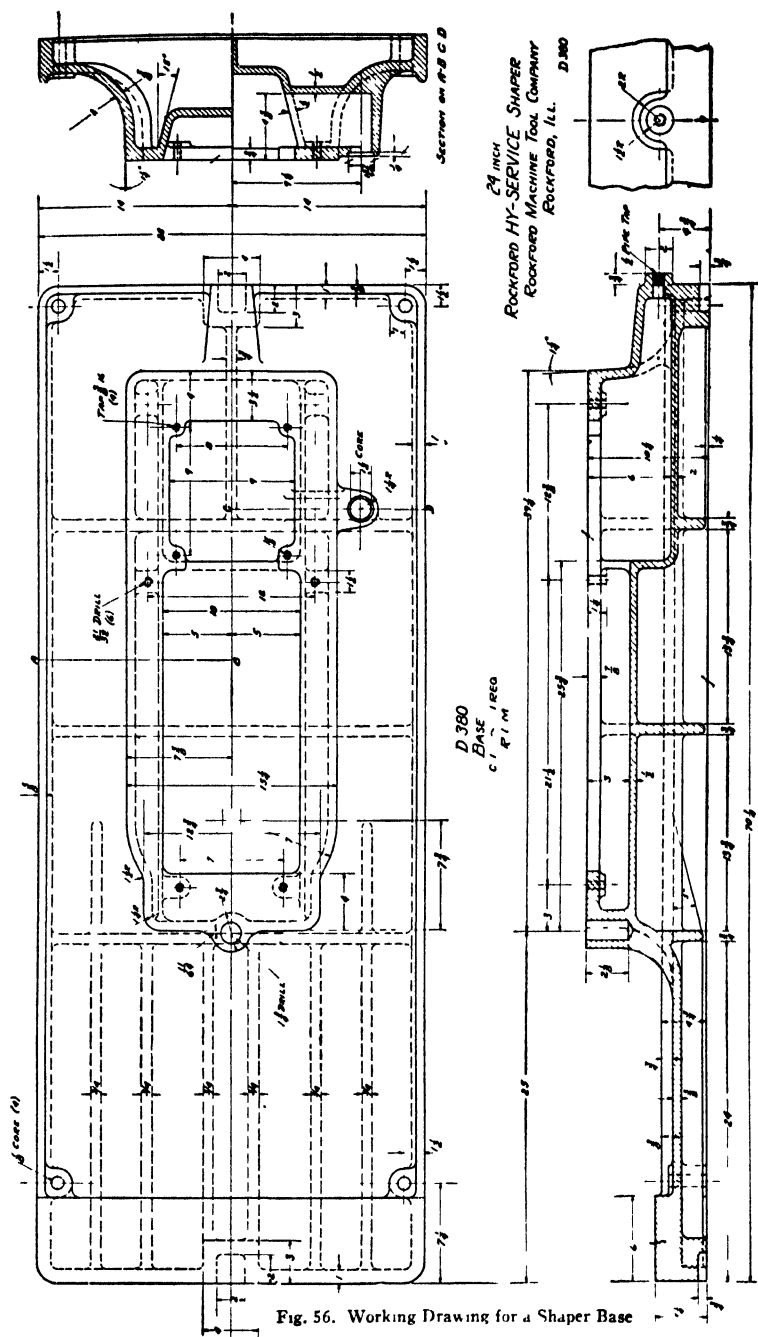
In the plan view near the front side of the upper part of the base, a $1\frac{1}{2}$ " cored hole is specified. By referring to the section *ABCD*, the pattern maker finds that this part of the base is connected to a perpendicular rib below it.

To the right in the end sectional elevation *ABCD* are two horizontal lines which outline the top and bottom of another part of the base. This part of the base is also shown in the front elevation and extends from the rounded corner of the lower part of the base to the mass of metal that contains the $1\frac{3}{4}$ " hole $2\frac{1}{4}$ " deep. The thickness of this part is $\frac{1}{2}$ ". This part of the base must be reinforced in some way and so the pattern maker looks for a vertical rib. In studying the front sectional elevation, he finds the vertical rib and the dimension $\frac{3}{4}$ ", which is the thickness of the rib. Then on the plan view, he finds that the rib runs practically the full width of the base. To the left and to the right of the vertical rib are ribs which are rather shallow, and from the front sectional elevation, the pattern maker finds that they are the same thickness as the center rib and from the plan view he finds that these ribs are the same length as the center rib. The pattern maker now studies the front sectional elevation and finds to the right of the left rib a tapered line making an angle of 15 degrees. He projects the tapered line up to the plan view by laying a rule or a machinist's scale on the blueprint at the end of the tapered line and finds the two ribs which taper at the bottom of the casting.

At the left of the drawing in the front sectional elevation is a visible horizontal line that is parallel to the bottom base line and tangent to the bottom of one of the shallow ribs which joins the left end or wall of the casting. This line shows the depth of the four long and two shorter $\frac{3}{4}$ " ribs which are shown in the plan view.

On the right end of the plan view is a specification— $1\frac{1}{2}$ " core. To find the depth of this hole, the pattern maker refers to the right end sectional elevation and finds it is to be $4\frac{5}{8}$ ".

The pattern maker again refers to the end sectional elevation in order to learn more about the details of this job. He looks at the plan view for a cutting plane line which would indicate where a section is taken. Referring to the front half of the plan view at the right end directly in front of the outside of the base, he finds a letter *D*. Following the perpendicu-



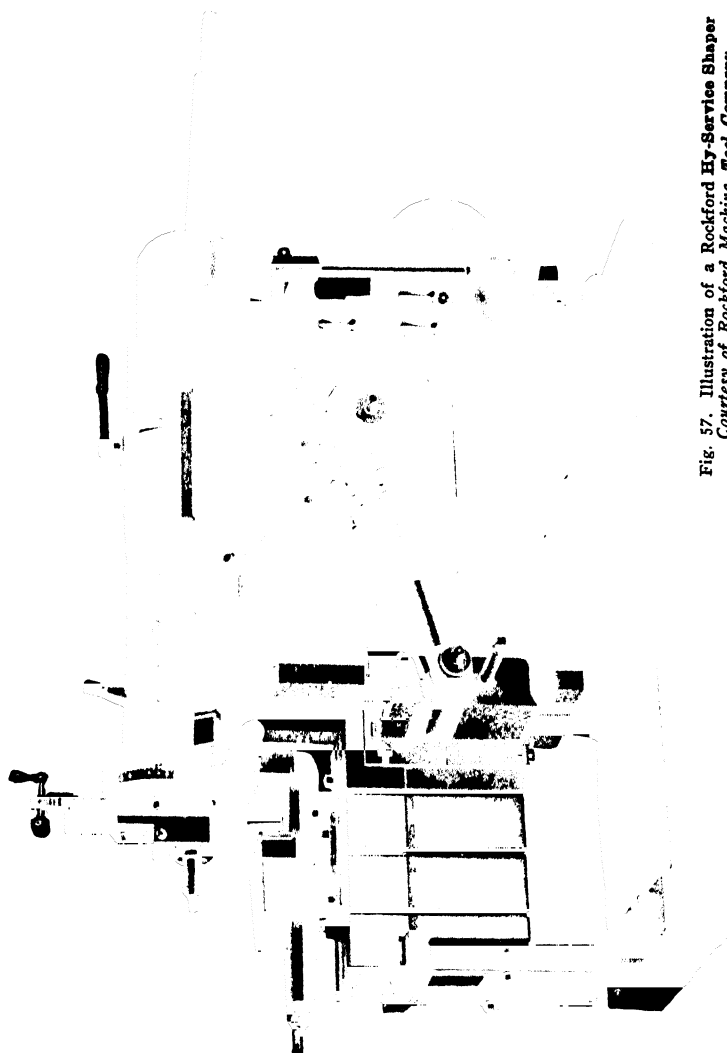


Fig. 57. Illustration of a Rockford Hy-Service Shaper
Courtesy of Rockford Machine Tool Company

lar line, which is the cutting plane line, up to the center line, he finds a letter *C*. This is only one-half of the width of the base; and as the end sectional elevation shows the full width in section, he looks for another cutting plane line. Following the horizontal center line to the left of the cutting plane line *DC*, he finds a short distance from *C* a letter *B*. Above the letter *B*, he finds the other cutting plane line with the letter *A*. Starting with letter *A*, the combination of cutting plane lines is called cutting plane *ABCD*. If the base were cut on these lines and the piece to the right that was separated from the left piece was removed, thereby enabling one to look at the base from the right end, a view would be obtained the same as shown at section *ABCD*. By a study of this section, the pattern maker is able to determine where some of the ribs and other parts of the casting are located.

Let us now study the blueprint from the point of view of the machinist. In the plan view on the center line of the base, he finds a hole $1\frac{1}{2}"$, so he looks for the specification which tells him whether the hole is to be cored, drilled, or reamed. $1\frac{1}{2}"$ drill is specified. Now the machinist wants to know whether this hole is to be drilled entirely through the base or just how deep it is to be. In reading the front sectional elevation, he finds that this hole is to be drilled to a depth of $2\frac{1}{4}"$.

At the extreme right end of the front sectional elevation is a tapered tapped hole. The machinist studies the blueprint to find the size of the tap to use and the length of the part tapped. Above the hole he finds the length is to be $\frac{1}{2}"$ and to the right the size of tap is to be $\frac{1}{2}"$ pipe tap. Referring to the right side or the right end of the base, he finds that the tapped hole is to be $4\frac{1}{2}"$ from the bottom of the base.

In the plan view the specification, tap $\frac{1}{2}$ -16 and directly below it (4), tells the machinist that four holes are required. Back of the horizontal center line at the extreme right end of the upper part of the base, he finds a dimension $4"$, which tells him that the first tapped hole is to be $4"$ from the right end. At the left of this dimension he finds that there are two tapped holes located back of the center line and are $9"$ apart. An experienced mechanic, familiar with blueprints, will readily understand that the front holes, not having locating dimensions, are to be laid out exactly in the same way as the back pair of holes.

Near the right pair of holes, he finds that they are to be spaced 8" apart. As no dimension is specified for the left pair of holes, it is understood that they are to be on 8" centers. The next step is to locate the holes in relation to the center line. On the blueprint he finds a dimension stating the center distance of the holes. As no dimension is indicated for the location of the holes from the center line of the base, it is understood that they should be equidistant, or 4" from the center line.

QUIZ QUESTIONS*

1. What is the over-all length and width of the base?
2. How is the surface at the left end of the base finished?
3. What is the width of this surface?
4. What is the width or thickness of the various ribs which are on the lower part of the base at the left end?
5. What is the width of the base on which the main part of the machine rests?
6. What is the diameter and depth of the hole that is located in the left end of the raised section of the base?
7. What is the diameter and depth of the cored hole which is shown just to the right of one of the main ribs in the plan view?

*No written answers are required for Quiz Questions.

MANUFACTURING DRAWINGS

Worm and Worm Wheel. In making a study of the blueprint of the worm and worm wheel, Fig. 58, it is readily seen that in so far as the principles of the drawing are concerned, they are the same as for the drawings used in machine building, however, the specifications vary to some extent as they apply to manufacturing conditions. This blueprint, Fig. 58, is a typical working drawing used in a manufacturing plant. It is also used by concerns that specialize in manufacturing gears, as will be understood from the note in the lower left-hand corner of the blueprint.

The blueprint shows two views—a front elevation and a right side elevation. It is easy to read, for the drawings are simple working drawings.

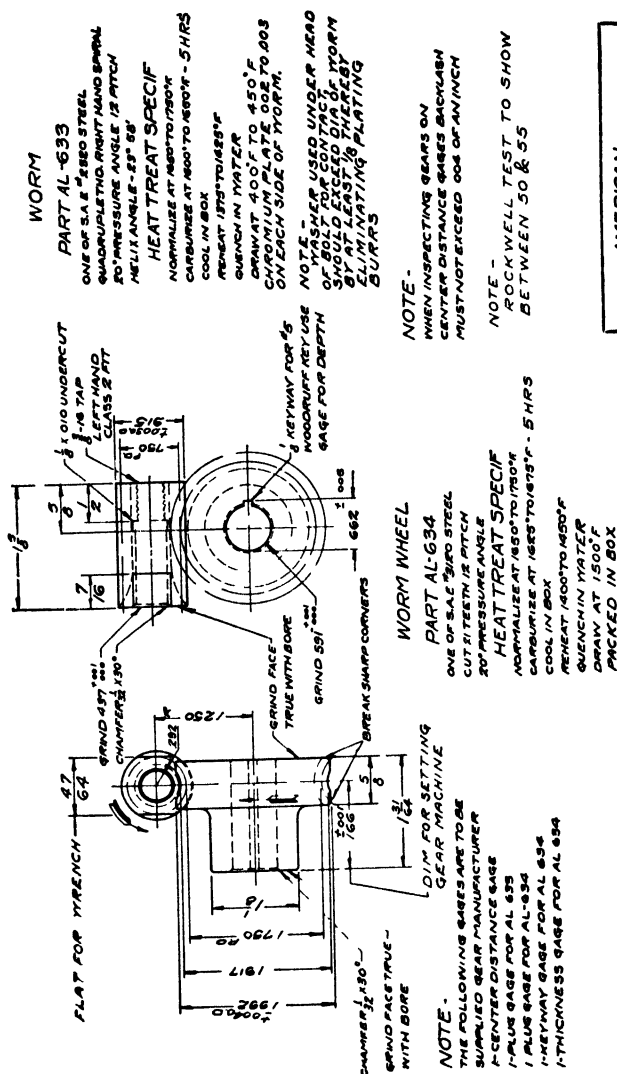


Fig 58 Working Drawing of
a Worm and Worm Wheel

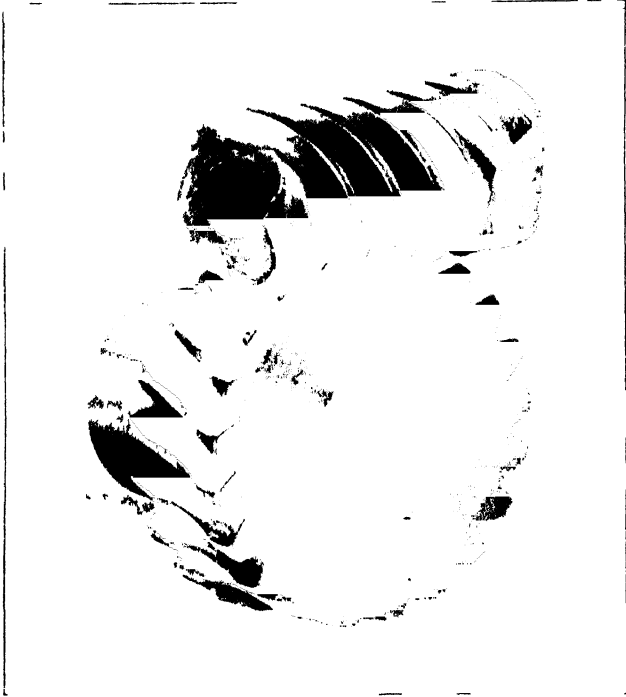


Fig 55A Illustration of Worm and Worm Wheel Shown in Fig 58
Courtesy of American Floor Surfacing Machine Company

Let us see just how the lathe hand would use this blueprint. We shall assume that he is going to machine the worm wheel. He must first find out what the finished outside diameter of the worm wheel is to be. At the left of the front elevation, he finds the dimension 1.992" plus and minus .004". Now he looks for the over-all length of the worm wheel. Referring to the front elevation, he finds it is to be $1\frac{3}{4}$ ". He next examines his blueprint to find the width of that part of the worm wheel on which the teeth are to be cut. He finds below the front elevation that the width is to be $\frac{5}{8}$ ". At the left of the front elevation, he finds the hub diameter is to be $1\frac{1}{8}$ ".

In the front elevation of the worm wheel, the lathe hand finds four invisible lines practically the same length as the over-all length of the worm wheel. These lines indicate that there is a hole and a keyway in this worm wheel, but they do not show whether the hole is round or square. Referring to the right end elevation, he finds that the hole is round and that the final operation on it is to grind the hole to a diameter of .591" plus .001" and minus .000". This dimension shows him that the gauge to be used in this hole, it being a plug gauge, is approximately the same diameter as the finished round hole.

The machine operator knows that the keyway runs about the full length of the worm wheel and he must find how wide and how deep this keyway should be. Referring to the right end elevation he finds the specification— $\frac{1}{8}$ " keyway for No. 5 Woodruff key. (See table at end of text.) Use the gauge for depth. This specification is for a piece being produced in quantities or on a manufacturing basis. To check a gauge for this purpose, compare the measurement with that below the right end elevation of the worm wheel; and if the gauge measures .662" approximately, there would be no question about it being the correct one.

Again the lathe hand studies the front elevation of the worm wheel and, on the left end or inside of the hub, he finds that there is a chamfer to be machined. For the size of this chamfer, he refers to the specification to the left of the drawing. This chamfer or countersink as specified on the drawing is to be $\frac{1}{16}$ " x 30°, indicating that the depth of this chamfer or countersink is to be $\frac{1}{32}$ " and the angle 30°.

There is a question regarding how accurate the hub is to be machined, therefore the lathe hand carefully examines the specifications near the drawing and finds the following note: *Grind face true with bore*. This indicates that the end of the hub is not only machined on the lathe but also on a grinding machine. Further study shows that the other end of the worm wheel must also be ground true with the bore.

The operator on the gear cutter is concerned about certain dimensions for setting up his job. He looks for the size of the hole in the worm wheel, also the distance from the top of the table to the center of his cutter. In the right end elevation, he finds that a line and an arrowhead point to the hole and the specification is .591". This tells him the proper size of mandrel* to be slipped in the bore of the gear. The distance from the table of the machine to the center of the gear face is 1.166" plus and minus .001".

Referring to the front elevation, the operator finds at the right of it the dimension 1.250", which is the dimension for locating the cutter in the proper relation with the worm wheel face. Setting to 1.250" enables him to cut the proper depth of teeth. On some blueprints 1.250" would be specified on the end elevation.

In considering the machining of the worm, the blueprint would first be studied in an endeavor to determine the finished sizes. If the lathe hand were interested in knowing the material that was used, it would only be necessary to refer to the specification list at the right of the blueprint under the heading *Worm*. In the first line he would find that the material was No. 2320 S.A.E., or Society of Automotive Engineers standard steel, and in this specification he would also learn that there was only one worm required for each machine.

The lathe hand now studies the blueprint to find the over-all length of this worm. At the top of the right end elevation, he finds that the over-all length is $1\frac{1}{2}$ ". The two horizontal invisible lines directly above and below the center line of the worm suggest to him that there is a hole of some kind in the worm. From the front elevation, he learns that the hole is round. To find the size of this hole, he refers to the left of the right end elevation and finds the specification—Grind

*A mandrel is a true bar on which gears, pulleys, etc., are mounted while being machined.

.437" plus .001" and minus .000". Very near the left end of the worm is an invisible perpendicular line and at each end of this line a short tapered line. These lines show that the end of the worm is slightly countersunk or chamfered. The lathe hand now looks for the size of the chamfer. At the left of the right end elevation, he finds the specification—Chamfer $\frac{1}{32}$ " x 30°. On the right end of the worm, he finds some invisible lines which he thinks are used to indicate threads. In making a more careful study of the blueprint, he finds that there are to be threads $\frac{3}{8}$ " in diameter with a pitch of $\frac{1}{16}$ ", or 16 threads per inch, tapped into the end.

Above and below the center line is an invisible line which starts from the right end of the worm and terminates $\frac{1}{16}$ " from the left end. The lathe hand studies the front elevation and, being familiar somewhat with worm gearing, decides that these lines indicate the bottom of the worm thread which is cut on the circumference.

From the drawings, the lathe hand is unable to determine whether the worm is to be single, double, or triple threaded, etc. So he refers to the specifications for the worm on the right side of the blueprint. Here he finds that the thread cut on the worm is to be quadruple with a right-hand spiral. He also finds in the specifications that the helix angle of the worm thread is to be 23° 58".

Suppose that the heat treating department is issued a blueprint, as shown in Fig. 58, with an order to proceed with the necessary heat treating processes for the worm. The heat treater must first find at what temperature to normalize the worm. He studies the specifications relative to heat treating and finds the specification—Normalize at 1650° to 1750° F. Then he looks for the temperature to carburize the worm and he finds the specification—Carburize at 1600° to 1650° F.-5 Hrs.

After the various heat treating processes have been completed, an order and a print is received by the plating department. The plater knows that the worm is to be chromium plated, but he must find just how thick this plating is to be. Referring to the specifications, he finds this note—Chromium plate .002" to .003" on each side of the worm.

In the inspection department, the inspector knows that the center distance must be gauged. To find how much allow-

ance can be made for backlash, he examines his blueprint and finds in a note in the lower right-hand corner the following information—When inspecting gears on center distance gauges, backlash must not exceed .004".

The worm gearing job being a manufacturing one readily suggests to a competent mechanic that certain commercial variations are allowed. These are not specified in decimals and common fractions on this blueprint as on some, but are provided for in a note at the bottom of the blueprint—Fractional dimensions on finished surfaces can vary plus or minus $\frac{1}{64}$ ".

QUIZ QUESTIONS*

1. What is the finished outside diameter of the worm wheel?
2. How is the hole in the worm wheel to be machined, and what is the size of this hole?
3. What size is the chamfer to be machined on the worm wheel?
4. What is the over-all length of the worm?
5. What is the size of the hole in the worm, and how is it machined?
6. What type of a thread is to be cut on the worm?
7. What material is the worm to be made of?

*No written answers are required for Quiz Questions.

Sanderplane. Fig. 59 is a blueprint of an assembly drawing of a sanderplane, showing a side view. A sectional elevation, taken through the pulley, worm wheel, and ball bearings, is shown in Fig. 60. A partial section indicates the worm, radial ball bearings, etc., in Fig. 59. Fig. 61 is a picture of a sanderplane.

The worm gearing, Fig. 58, is a mechanism used in a sanderplane. There might be a question from one of the mechanics in the assembly department as to where the worm gearing mechanism is to be located in this machine. On a standard machine, this question would not be necessary, however, there is a possibility of a question of this kind arising with a new machine, that is, the first one to be built. A study of the assembly drawing explains the location of the worm gearing.

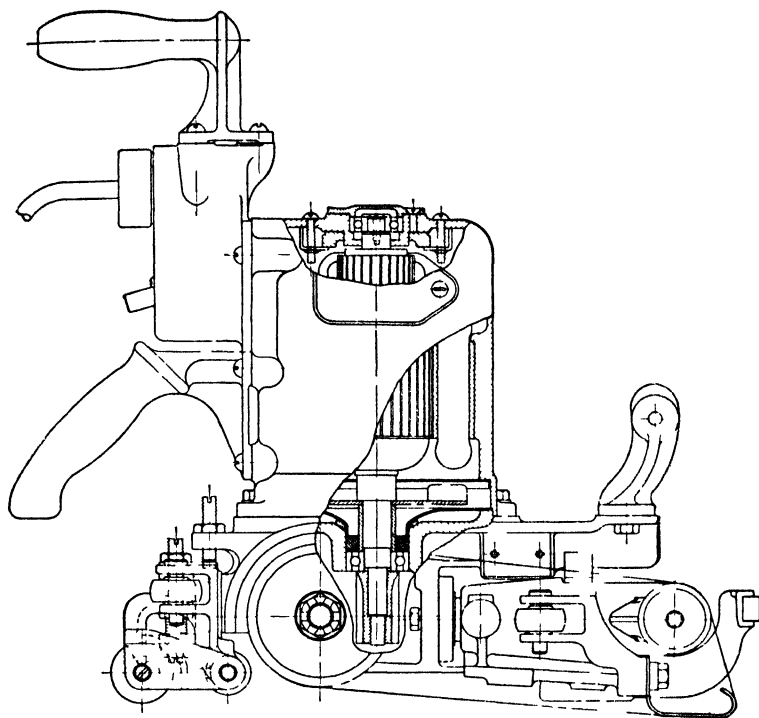


FIG. 59 Illustration of an American Sanderline
 Courtesy of American Floor Surfacing Machine Company

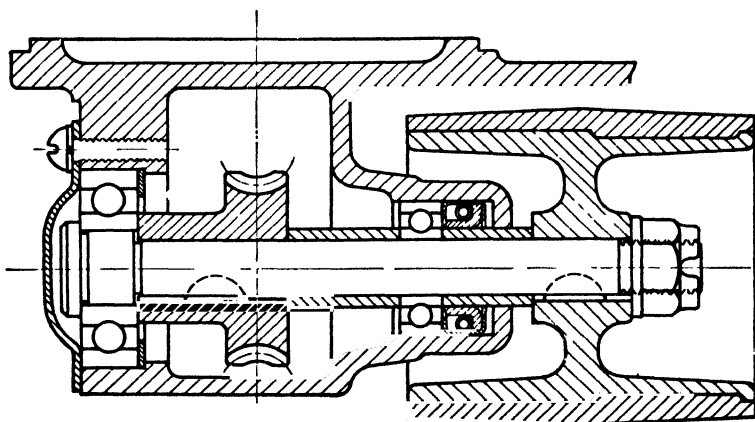


Fig. 60. Section through Pulley, Worm Wheel, and Ball Bearing on a Sanderline
 Courtesy of American Floor Surfacing Machine Company

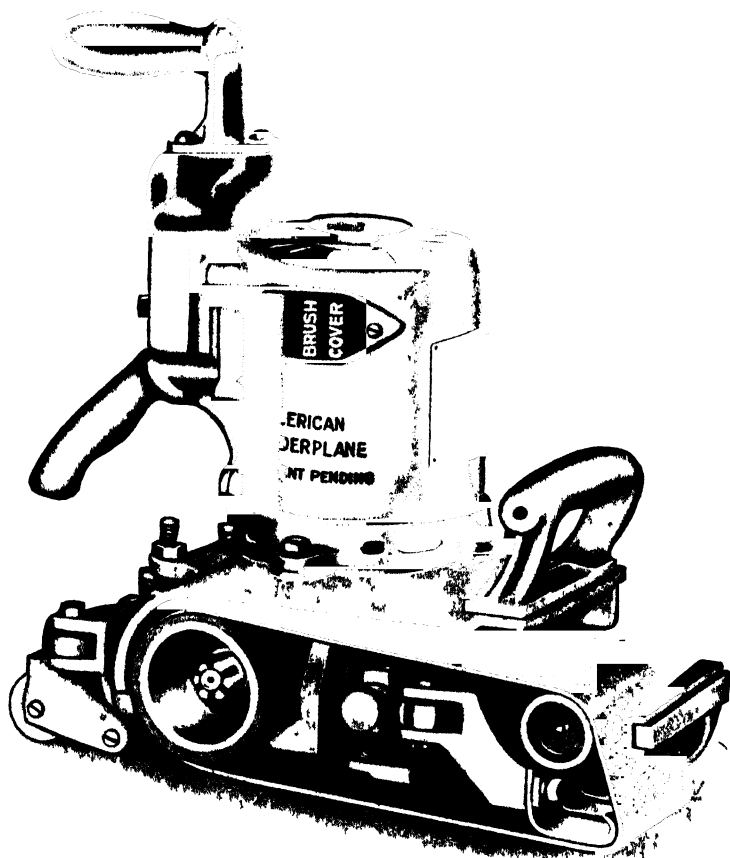


Fig. 61 Illustration of a Sanderplane
Courtesy of American Floor Surfacing Machine Company

The mechanic knows that the upper part of the sander-plane consists of the motor end bell, motor, ball bearings, etc. He studies the assembly drawing carefully and finds near the central part of the machine that the worm screws on the end of the spindle. He compares it with the worm on the working drawing, Fig. 58, and finds it is the same. The working drawing specifies a $\frac{3}{8}$ "-16-tapped thread cut to a depth of $\frac{1}{2}$ ", also a counterbored hole back of it. The worm wheel is found in the sectional elevation.

The mechanic finds from a study of the sectional elevation, Fig. 60, that a radial ball bearing is first assembled on the spindle, then the worm wheel comes next, or directly to the right side of the ball bearing. Near the other end of the spindle, he finds another radial ball bearing, which is seen in the sectional elevation. A cylindrical separator is used between the worm wheel and the ball bearing shown at the right. To the right of this ball bearing and to the left of the pulley hub, another spacer is found. To the right of the pulley hub, a washer is shown and on the outside of the washer is a hexagonal nut.

Before the mechanic examines the assembly drawing for assembling the worm wheel to the spindle, he must find out the type of key to be used. He finds this information on the working drawing, Fig. 58.

Figs. 62-A and 62-B are blueprints of the motor end bell. Four views are shown—a front elevation, a plan view, a sectional plan view, and a sectional right side elevation.

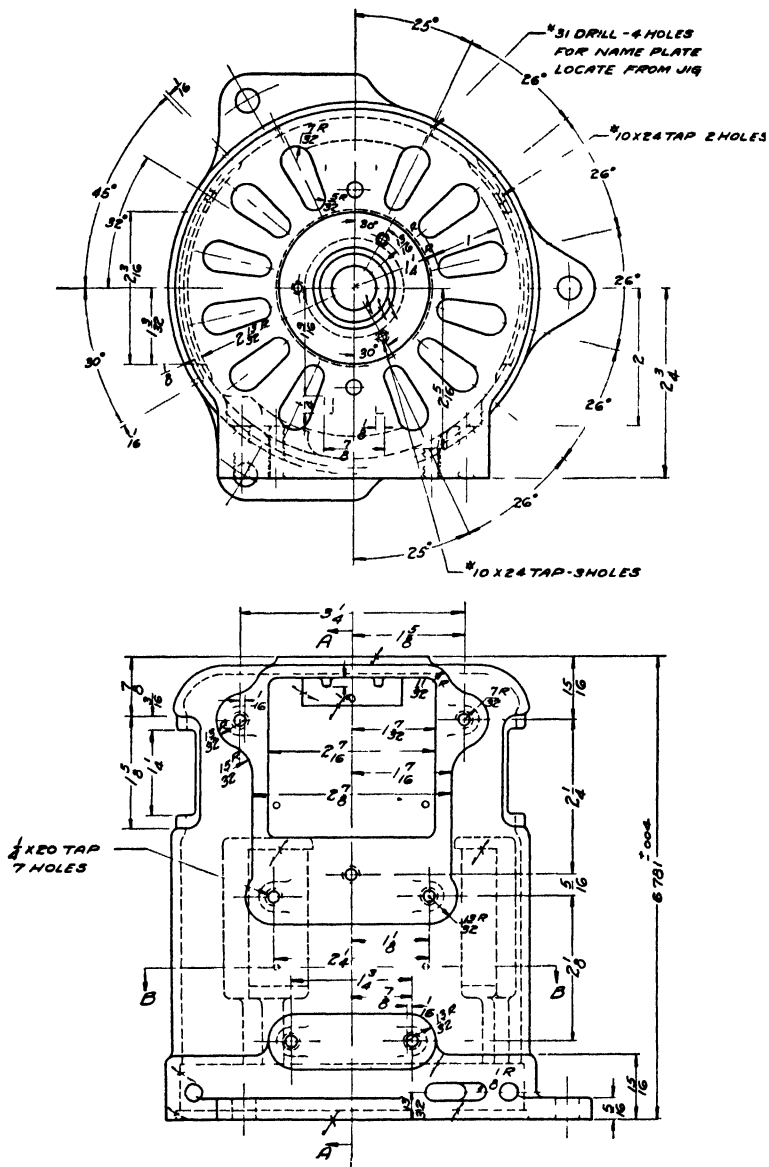
In the upper part and on the sides of the front elevation the horizontal invisible line terminating into curves and the two perpendicular invisible lines on the sides indicate the inside surface of the end bell, or, in other words, the inside of the wall of the bell.

In reading the front elevation, the pattern maker must find out whether the casting is to be round or square or just what shape it is to be. He studies the plan view and the sectional plan view and finds that the end bell is to be cylindrical. Now he wants to know the thickness of the wall of the bell. From the sectional elevation AA, which is taken on the center line of the front elevation, he finds the thickness of the wall is to be $\frac{1}{8}$ ". He next wants to know the over-all height of the end bell. Referring to the blueprint, he finds to the right of

the front elevation the dimension 6.781" plus and minus .004", which is the over-all height of the end bell.

A question comes up about the openings in the top of the end bell and what the sizes are. To get these dimensions it is necessary for the pattern maker to study the plan view of the end bell. At the right of the perpendicular center line in the first quadrant, he finds the openings are specified as being 1" long and rounded at the ends. He now proceeds to determine what radii the ends should be. In the second quadrant, or that part of the drawing to the left of the perpendicular center line, he finds that the larger radii is to be $3\frac{1}{2}"$ and the smaller one $\frac{1}{2}"$. In the first quadrant near the horizontal center line he finds a dimension $1\frac{1}{4}" R$ (radius) which shows how near the edge of the end bell or how near the center these openings should come. He must know how to space these openings. He finds on the blueprint outside of the plan view in the first quadrant a specification 25° . This locates the first opening and the figures on the outside of the end bell and to the right show the other openings are to be 26° apart between center lines.

In examining the plan view, the pattern maker finds that the front part of the object has four tapped holes in it. Referring to the front elevation, he finds that the projection is a pad having four tapped holes in it. At the upper part of the pad, he finds the outline of a square and as the plan view does not indicate a raised part on the projection, it would naturally follow that the square must either be an impression of some kind or a hole through the wall of the end bell. Referring to section AA, he finds there is a hole through the wall but that it is not square but a rectangle $2\frac{1}{2}"$ long and $2\frac{7}{8}"$ wide, as indicated on the front elevation and sectional elevation. The pattern maker now looks for the width of the pad. At the center of this pad, he finds the dimension $2\frac{7}{8}"$, but he does not find a direct dimension for the length, so he must do some calculating. He finds the sum of $\frac{1}{8}"$, $2\frac{1}{4}"$, $\frac{1}{8}"$, and $\frac{1}{2}"$. This gives him the distance from the bottom of the pad to the top of the end bell, which is slightly more than the length of the pad. As some deduction must be made, he refers to the sectional elevation AA and finds that the boss on the top of the end bell is $\frac{1}{8}"$ high, this being the dimension then to be subtracted from the sum just obtained.



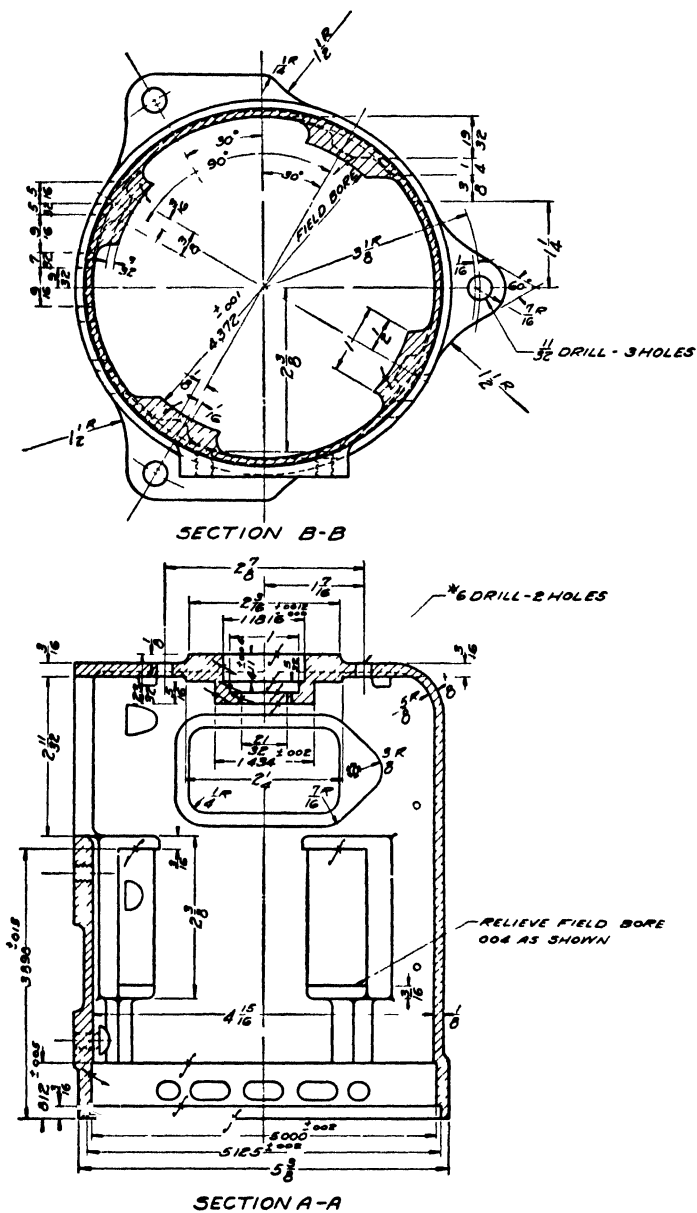


Fig. 62-B. Working Drawing for an End Bell Showing Front Sectional Elevation *AA* and Top Section *BB*

Below the upper pad in the front elevation, the pattern maker finds an impression or a pad which is of a rectangular design, and as there are two tapped holes in it, it is more likely a pad than an impression in the front of the motor end bell. The sectional plan view *BB*, which is taken on the cutting plane line *BB* shown in the front elevation, convinces him that the two horizontal lines terminating in arcs is a pad. The sectional elevation *AA* also explains this. To find out how far this pad and the upper one projects out from the end bell, the pattern maker examines the plan view and directly to the right of it finds the dimension $2\frac{3}{4}$ ".

In the upper part of section *AA* intersecting the center line there is a rectangular shape, the one end tapering into a $\frac{3}{8}$ " arc. At the right of the end of the rectangle is a tapped hole, which suggests that a pad of some sort must be in the inside of the cylinder which is shown clearly in the plan view. On the left side of the plan view, the pattern maker finds two very short horizontal lines above and below the horizontal center line, indicating that there is an opening in the wall of the end bell. Now he must find out what the length of this opening is to be and how far one side is to be located from the horizontal center line. At the left side of the plan view, he finds the dimension $2\frac{3}{16}$ ", which is the length of the hole, and he also finds that one side of the hole is to be located $1\frac{3}{32}$ " from the center of the end bell.

The lathe operator is now ready to proceed in machining the casting. We shall, therefore, consider what he is concerned about. Both ends of the end bell are to be faced, therefore, he must know what length to machine the casting. To the right of the front elevation, he finds the dimension 6.781".

He knows he has some machining to do in the way of counterboring for a ball bearing, etc., but he does not know what diameter or depth to make the counterbores, therefore, he refers to the upper part of the end bell in section *AA*, and finds that the first counterbore is to be machined to a diameter of 1.1816" plus .0012" and minus .000" and to a depth of .411" plus and minus .004". Below this counterbore is a second counterbore with a diameter of 1" and depth of $\frac{5}{32}$ ". Below the second counterbore is a through hole. The machinist finds the specification $\frac{3}{16}$ ", which is the diameter this hole is to be bored out to.

Inside of the upper part of the casting in which the counterbores and through hole are machined, a turning (machining) operation must be performed. The lathe operator must find out what the diameter of this part of the casting is to be. From the blueprint, he finds the diameter is to be 1.434" plus and minus .002". As gauges are used for gauging the counterbores, etc., the mechanic must make sure that he has received the proper gauges for this job.

The machine operator is ready to bore that part of the end bell in which the field magnet of the motor fits. He looks for the diameter of the bore. In section *BB*, he finds that in the first and third quadrant, projections are indicated and the interior surface has a finish mark *f*, which apparently pertains to the bore for the field, and the dimension 4.372" plus and minus .001", which is the diameter of the bore. In the second and fourth quadrant are also shown projections. The mechanic, being an experienced blueprint reader, understands that the dimension between these two projections is the same as for the other two referred to. In manufacturing practice, the machine operator uses a gauge for gauging this bore. This information, however, does not suffice for the machine operator's job. He looks for a dimension which will give him the distance from the bottom of the end bell to the top of the bore for the field. From his experience he knows that he will find this dimension either on the section *AA*, outside of it, or on the front elevation. At the left side of section *AA*, he finds this dimension, which is 3.898" plus and minus .015". For measuring this depth, he also uses a gauge.

Ready to counterbore the bottom of the end bell, the machine operator procures gauges of the proper sizes. He checks up to be sure that they are the right gauges by examining the dimensions on the blueprint. From the sectional elevation *AA*, he finds that the gauge for the first counterbore must be one that will gauge a diameter 5.125" plus and minus .002". His gauge for the second counterbore must be one that will gauge a dimension 5.000" plus and minus .002" satisfactorily.

QUIZ QUESTIONS*

1. What is the thickness of the wall of the end bell?
2. What is the over-all height of the end bell?

3. What are the sizes of the openings in the top of the end bell?
4. What is the dimension for boring out that part of the end bell in which the field fits?
5. What are the dimensions given for the rectangular shape shown in the upper part of section AA?
6. Are both ends of the bell to be faced?
7. What dimension must the lathe operator use in doing this job?

*No written answers are required for Quiz Questions.

TOOL DRAWINGS

Some years ago piece parts were not produced in the quantities they are today, and the experience gained from efficiency methods has led up-to-date manufacturing concerns to readily realize the importance of blueprints for the tool maker and punch and die maker.

For simple jigs it is not absolutely necessary to make detail working drawings. Dimensions, notes, etc., are specified on an assembly drawing such as shown in Fig. 63. However, in comparatively complicated jigs and fixtures, detail working drawings have proved more satisfactory than the assembly drawing with all its dimensions, notes, etc.

In some tool rooms assembly drawings are used by the punch and die maker, depending on the complexity of the punch and die; while in other plants, even though a punch and die is complicated, it is the practice to use assembly drawings that show the necessary dimensions, specifications, etc.

Drill and Tap Jig. A tool maker would be given a blueprint showing the drill and tap jig, Fig. 63, also the piece part that was to be drilled and tapped, Fig. 64. Here is shown only a working drawing of the piece part. Fig. 64 shows three views—a plan view, a front elevation, and one plan view, taken at an angle, of a packer link bearing. The title on the blueprint of the jig states that it is to be used for drilling and tapping a hole that is to be subsequently used for an oil cup, on the end of which would be found a $\frac{1}{8}$ " pipe thread. The tool maker must find the location of the tapped hole. Referring to the plan view, he finds on the center line a tapped hole is indicated and near the hole is the specification— $\frac{1}{8}$ " pipe tap. To the right of the plan view, he finds that the hole,

even though located on the center line, is not equally spaced from the ends of the bearings. From the far end of the bearing to the center line, he finds that the hole is to be at a distance of $2\frac{1}{4}$ " and from the other end $2\frac{3}{8}$ ".

Now, try to picture in your mind from a study of this blueprint just how the finished piece part appears, as this will make the reading of the tool drawing simpler. To enable the tool maker to better understand just how the piece part is to fit into the jig, the tool designer indicates the piece part with dotted or lighter lines so that there is a distinction between the piece part and the jig itself. This is not always the case with tool drawings as sometimes the tool maker has only the piece part and from it must determine just how it would fit into the jig.

From an examination of the plan view, Fig. 63, the tool maker gets a fairly good understanding of how the piece part is located in the jig and also how it is held. Referring to the right side elevation, he finds that the bearings of the piece part are pushed on a pin up to a shoulder, this part of the pin appearing to be like a washer. On the plan view, the dimensions of this so-called washer are found to be $\frac{3}{16}$ " thick and $1\frac{1}{2}$ " in diameter. In the upper part of the right side elevation, he finds that pin No. 9 is used to be inserted in the .635" diameter hole of the third packer link bearing. This pin as will be seen from the plan view passes through the one hole only.

The tool maker knows a bracket, Detail No. 5, Fig. 63, is required for this job. On the plan view, he finds that Detail No. 5 is the bushing plate and that it is held down on the base plate by a hexagonal cap screw and two dowel pins. The bushings are located on the left end of the bushing plate or above that part of the third packer link bearing that is to be drilled and tapped. Referring to the front elevation of the bushing plate, he finds that there are two bushings to be used for this job, the one being inserted in the bushing plate and the second bushing being inserted in the bushing that fits in the bushing plate. One bushing is used for a drilling operation and the other for a tapping operation.

The tool maker, having received the casting for the base plate, knows fairly well the design of this piece, but he must study his blueprint in order to get the necessary information for the machining processes on this job. On the front and the

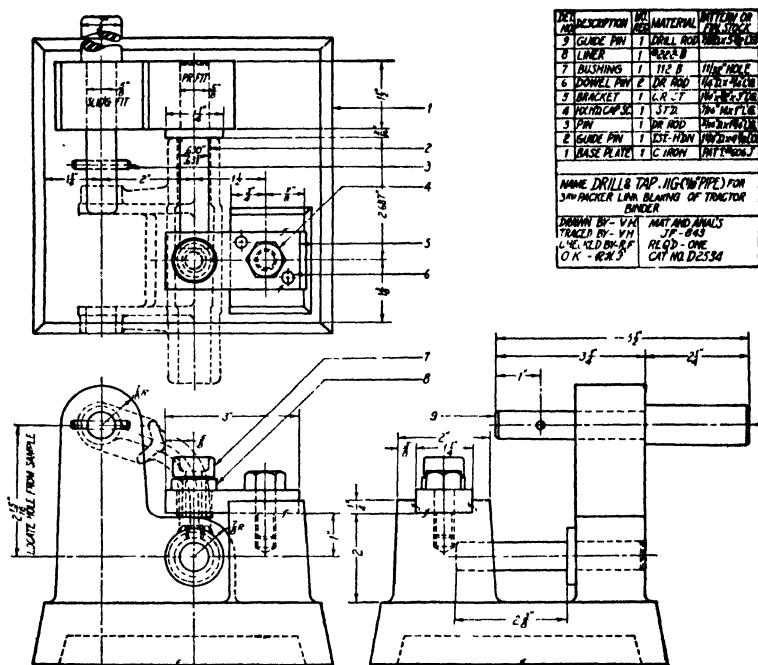


Fig. 63. Working Drawing for a Drill and Tap Jig

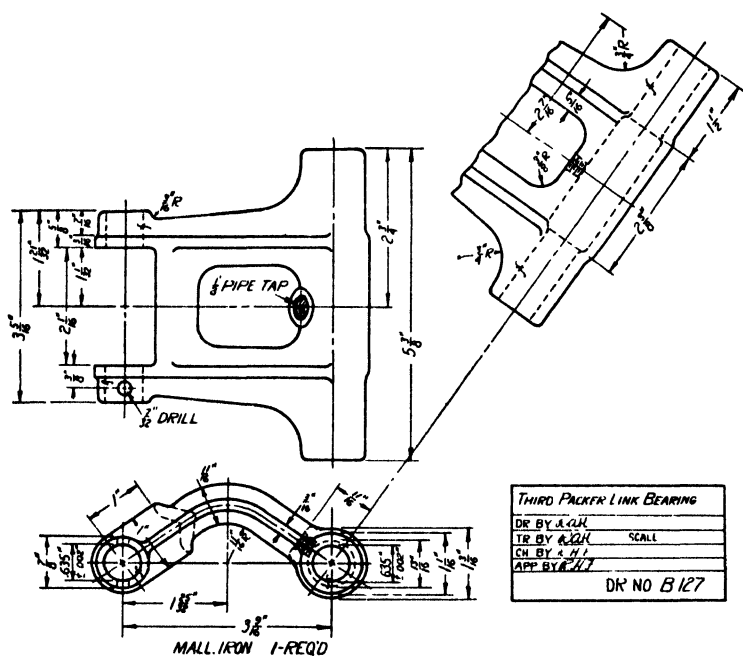


Fig. 64. Working Drawing for a Packer Link Bearing

right side elevation at the bottom of the drawing, he finds the symbol *f*, which means that this part of the base is to be machined. He completes this part of the job. Now he looks for the necessary information for the operation that is to be performed on the shaper, which is the upper part of the base in which the bushing plate fits. At the left of the right side elevation, the dimension $\frac{1}{4}$ " shows him that he must machine to this depth to correctly locate the bushing plate. From the right side elevation, he also finds that the width of slot for the bushing plate is to be $1\frac{1}{4}$ " wide and is to be located $\frac{3}{8}$ " from the left edge of the base plate.

In studying the right side elevation, the mechanic finds that the lower pin Detail No. 2 fits into the base plate. To find what diameter the hole is to be machined to in which this pin is to fit, he refers to the plan view and, near the top, he finds the specification— $\frac{5}{8}$ " PR.Fit*. From this he knows the size reamer to use is $\frac{5}{8}$ ". He then makes the pin slightly larger in diameter to allow for the press required.

Knowing that the pin Detail No. 9 to the left of the one Detail No. 2 is the one used in the upper part of the jig, he refers to his blueprint to obtain the size of the hole in that part of the base plate. He finds the specification— $\frac{5}{8}$ " Slid'g Fit. From his experience, he knows it is advisable to use standard tools. He drills and reams this hole, using a $\frac{5}{8}$ " diameter reamer. The sliding fit then must be provided for on the pin or, in other words, the pin must be slightly less in diameter than the hole.

Referring to the plan view, he finds the various proportions of the guide pin No. 2, but he is unable to obtain the length of that part of the pin that fits in the piece part. On the right side elevation at the bottom of the drawing, he finds the length is to be $2\frac{3}{4}$ ". On the plan view, he finds the diameter of this part of the guide pin is to be $\frac{.630}{.631}$ "

In studying the bushing plate Detail No. 5, the tool maker finds the length is to be 3" and the width $1\frac{1}{4}$ ", but the thickness of this plate is not given. In some plants, bushing plates of various sizes are kept in stock, being standard, and can be used for other jigs. All that is necessary for the tool maker to do is to procure a bushing plate which calls for a size

*PR. Fit means—Press fit or a fit requiring pressure.

3"x 1 $\frac{1}{4}$ ". He finds that the thickness of this bushing plate is $\frac{1}{2}$ ".

No dimensions are specified for the location of the dowel pins nor for the location of the bushing and hexagon headed cap screw. The dimensions for these locations were not specified because, according to standard practice in the particular plant in which the jig was to be made, it was not necessary to specify the dimensions for those parts of the bushing plate.

For the proper materials that should be used for the various pieces of this jig, the tool maker refers to the bill of material in the upper right-hand corner or directly above the title. To find whether the guide pin No. 2 is to be made from machine steel or tool steel, he looks up the Detail No. 2 in the bill of material and in the fourth column, he finds the abbreviation—T. ST. H'DN. This means the guide pin is to be made from tool steel and hardened. The last column indicates that it is to be made from a piece of stock that is 1 $\frac{1}{4}$ " diameter by 4 $\frac{1}{8}$ " long.

QUIZ QUESTIONS*

1. What is the size of the hexagonal cap screw that holds down the bushing plate?
2. How far is the plate set into the base plate?
3. What is the dimension of pin Detail No. 2?
4. What is the size of the hole which the pin Detail No. 9 fits into?
5. What would the mechanic do in locating the position of the dowel pins, Detail No. 6?
6. What material is the base plate made of?

*No written answers are required for Quiz Questions.

Boring Jig. Another type of jig is shown in Fig. 65. As will be seen from the title on the assembly drawing, this jig is to be used for boring both ends of a connecting rod. Experienced tool makers are employed in the plant in which this jig is to be made, therefore it was only necessary for the draftsman to show two views for the assembly of the jig—a plan view and a front elevation.

For a job of this kind, the tool maker is given a sample connecting rod which not only shows him just what holes are

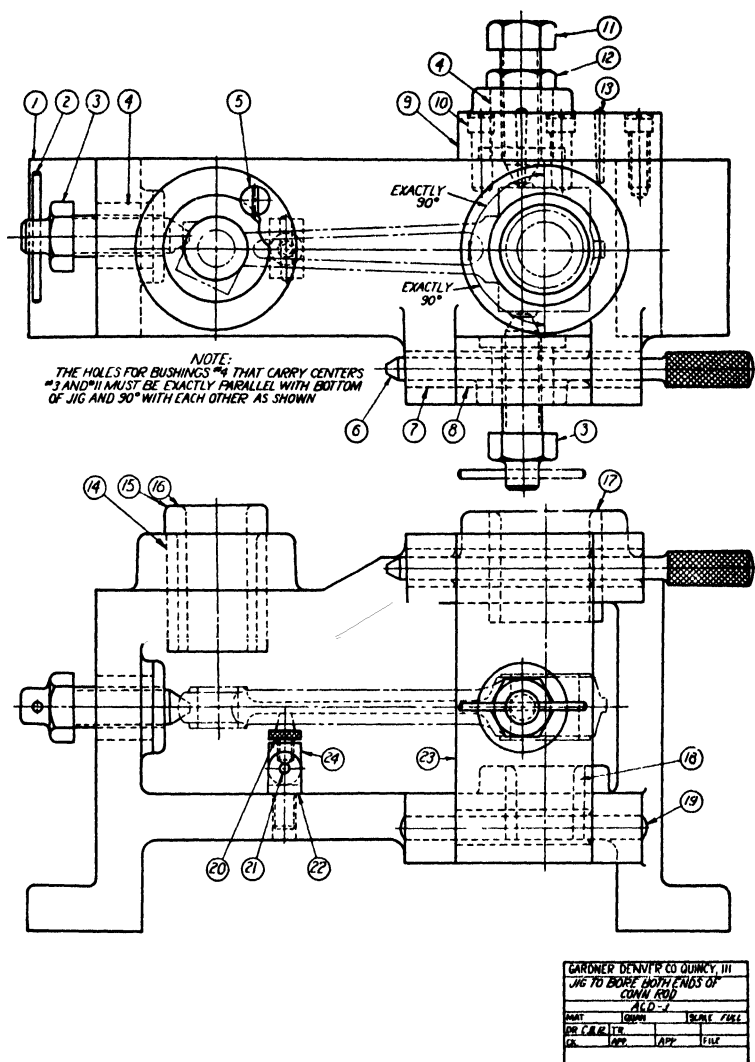


Fig. 65. Assembly Drawing for a Boring Jig

to be bored but also how to proceed with his job and just how the piece parts fit into the jig. To help the student to visualize the design of this piece and see just where the holes are to be bored in the connecting rod, Fig. 66 is given. Before proceeding to read the drawing of the boring jig, the student should make a study of the blueprint of the piece part, as this will help him to understand better how to read the blueprint of the jig.

From a study of the blueprint, Fig. 65, it will be seen in the plan view that the connecting rod is mounted on three tapered pins, the one being on the left end and the other two on the right end. The one on the left end, numbered 3, and the one at the front right, numbered 3, are of the same design. The back pin, or that one at the back right, is numbered 11 and the design of it is slightly different from that of the other two. To avoid springing the connecting rod, the designer found it necessary to use an adjustable stop or support, this being indicated very clearly in the front elevation near the left end. The adjustable part is number 20.

To remove the connecting rod from the jig, it is necessary for the machine operator to unscrew Detail No. 3, which is shown in the front elevation near the right end. Some adjustment will be necessary on Detail No. 3 at the left end of the jig. In addition to this, it is necessary to remove pin 6, which is knurled on the right end. This pin is pulled to the right, allowing a leaf in the front of the jig to drop down. This leaf hinges on pin 19. From a study of the left end of the front elevation, especially the upper part of it, two bushings are shown, one being numbered 14 and the other 16. Bushing 14 is pressed into the body of the jig, whereas, bushing 16 is a slip bushing and is kept from turning by the shoulder screw numbered 5. The plan view shows that this bushing is grooved out where the screw is shown, also that nearby it and between the two concave surfaces there is a flat spot. The purpose of this is to give the operator a chance to give the slip bushing a short turn and then, pulling it upward, to remove it for the second boring operation.

At the back of the jig at the top of the plan view is shown a plate numbered 9. Its purpose is to allow bushing 4 to be inserted into it. Bushing 4 is used for screwing in tapered pin 11. Plate 9, as will be seen from the plan view, is fastened

to the body of the jig by oval fillister head screws; and dowel pins are used to avoid any possibility of this plate shifting.

QUIZ QUESTIONS*

1. How did the designer avoid the tendency of springing the connecting rod when it was being machined in the jig?
2. How is the bushing Detail No. 16 kept from turning?
3. What is the size of the grooved-out part in bushing Detail No. 16?

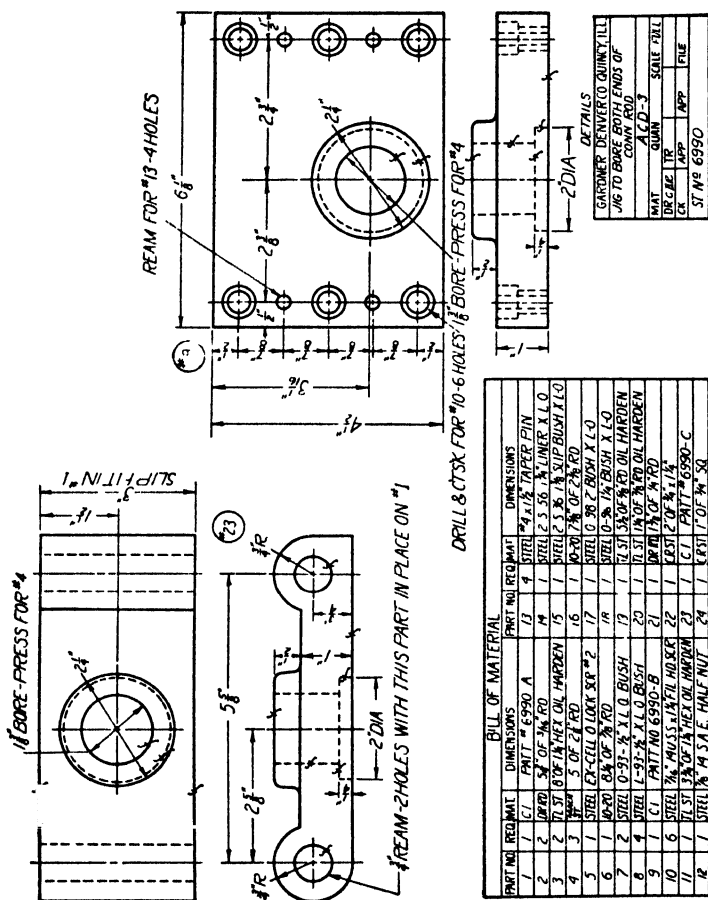
*No written answers are required for Quiz Questions.

Let us see just what machining operations are to be performed on the casting for the body of the jig. In starting out on this job, the tool maker might not know whether the bottom of the jig was to be machined. Therefore he refers to his blueprint, Fig. 67, and finds that finish marks are indicated at the bottom. This means that it is necessary to remove the scale on this part of the casting.

He again studies his blueprint to find out what other shaping operations are to be performed. He finds that that part of the jig in which the leaf or Detail No. 23 fits must be machined. He also finds that both top and bottom of the jig must be machined.

He now looks for the distance from the center of the $2\frac{1}{2}$ " hole to the inner face of the left ear. On the plan view, he finds that an imaginary line is located .500" plus .001" and minus .001" to the left from the center of the $2\frac{1}{2}$ " hole. To the left of the imaginary line between the ears, he finds the dimension $1\frac{1}{2}$ ". Therefore the sum of $1\frac{1}{2}$ " and .500" is the distance. Directly below the $1\frac{1}{2}$ " he finds a 3" dimension which is the distance between the ears. This dimension applies to the upper and lower ears for the leaf, or Detail No. 23, fits into the lower part of the jig as well as into the upper part.

In performing the operation in the drill press, the mechanic knows that holes are to be machined in the ears. From a study of the blueprint, he finds that the holes in the upper pair of ears are different than those in the lower pair. He gets this information from a detail of one of the ears. Near the left ear of the plan view is a specification — $\frac{3}{4}$ " ream pressed for #7. Near the lower left ear in the front elevation, a detail is shown and to the left is a specification — $\frac{1}{2}$ " ream



press for #19. We shall assume that the shaper hand has completed his work on the body or Detail No. 1. Therefore, the drill press hand proceeds in machining the holes in the piece that has been previously laid out.

QUIZ QUESTIONS*

1. What is the distance from the center of the $2\frac{1}{2}$ " hole to the inner face of the left ear?
- 2. How is the bottom of the jig finished?
3. What is the distance between the ears?

***No written answers are required for Quiz Questions.**

The blueprint, Fig. 68, has two Details—No. 9 and No. 23—each Detail showing a front elevation and a plan view. When the drill press operator receives the partly completed piece Detail No. 9, the $1\frac{3}{4}$ " hole in the piece and the 2" diameter by $\frac{1}{2}$ " deep counterbore have already been done on the lathe, therefore he proceeds immediately with the machining of the holes shown in the plan view. Drilling and counterboring for the oval fillister head screws are the first operations. So he must find out how far from the right edge the first hole is to be located and also what the distance is to be between the centers of these holes. On the plan view, he finds that the first hole is to be located a distance of $\frac{1}{2}$ " from the right edge. As he does not find direct dimensions for the location of the centers of the holes, he proceeds to find the sum of the two dimensions— $2\frac{3}{4}$ " and $2\frac{3}{8}$ " which gives the distance between centers of counterbored holes and dowel pin holes. Now he must find the perpendicular distance from the edge of Detail No. 9, which is $\frac{1}{2}$ ". Next he must find the distance between centers of counterbored holes and dowel pin holes, which is $\frac{1}{8}$ ".

QUIZ QUESTIONS*

1. On piece Detail No. 9, what is the size of the counterbored holes?
2. What is the size of the smaller holes?
3. Are these holes bored through the piece?

***No written answers are required for Quiz Questions.**

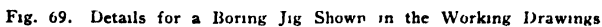
In making the bushings, the machine operator makes a study of his blueprint, Fig. 69, and finds the drawing is for two different pieces. The specifications are directly below the front elevation of the bushing drawing. One specification is—Make one .8125"; and the other is—Make one .875". The dimensions being specified in decimals show the mechanic that the holes must be accurate, therefore he must work to decimals rather than common fractions. The final machining process in the hole is a grinding one, indicated by the symbol *G* on an invisible line, which shows the side of the hole in the bushing. The final operation on the outside of the bushing is also a grinding one, as seen from the symbol *G*.

The head of the bushing shows that it is to be knurled, but no specification is given. This drawing is used in a plant by high-grade tool makers who are in a position to know just what sizes of knurls they have in stock or just what might be the standard practice in that particular tool room.

Directly below the head is a horizontal line, at the ends of which are two small arcs, indicating that part of the bushing is to be slightly concaved. This is to simplify the grinding process. Again no dimensions are specified and it is understood that the tool maker is in a position to determine for himself just what depth the grooving (necking) should be.

A mechanic not well experienced in reading drawings would encounter some difficulty with only a plan view and a front elevation of Detail No. 22, shown in Fig. 69. This drawing was used in a tool room by a tool maker who did not need another view. The tool maker studies the front elevation. The invisible line indicates that a slot is to be milled in the upper part of Detail No. 22. To make sure of this, he refers to the plan view and finds that two horizontal visible lines spaced a distance of $\frac{1}{16}$ " apart indicate there is to be a slot in this piece that is to be $\frac{1}{8}$ " wide and $\frac{1}{16}$ " deep, the $\frac{1}{16}$ " being obtained by subtracting the $\frac{1}{16}$ " dimension from the $\frac{1}{8}$ " dimension.

In making the tapered pin Detail No. 3, Fig. 69, the tool maker finds from the blueprint that it is to be made from hexagonal stock. He does not, however, find on the detail drawing, Fig. 69, what material is to be used. Therefore, he looks for the bill of material on Fig. 68, and finds in column 3 for part 3, or Detail No. 3, that the pin is to be made of tool steel. At the right of the front elevation of part 3, or Detail



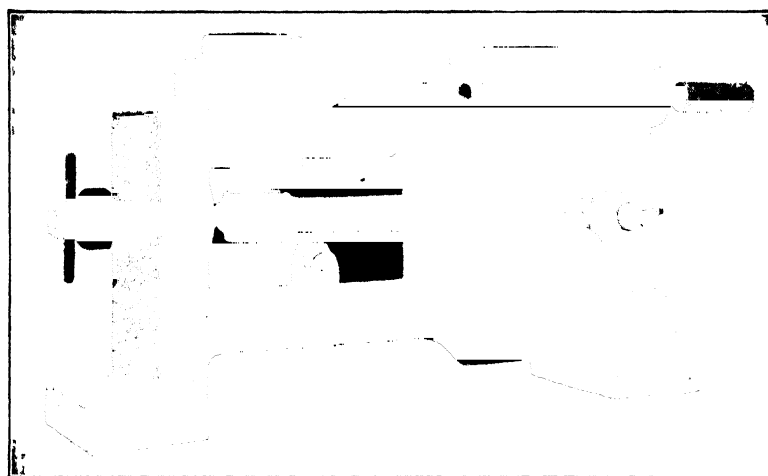


Fig 69A. Illustration of a Boring Jig Shown in Fig 65
Courtesy of Gardner-Denver Company

No. 3, Fig. 69, he finds the specifications—Press for No. 2. This indicates that this hole is to be drilled and reamed and that the piece that fits into it must fit in tightly. On the working drawing there is no dimension relative to drill and reamer size nor the size of No. 2 that fits into the hole. So he refers to the bill of material, Fig. 68, and in the fourth column for part 2, or Detail No. 2, he finds that the diameter of this part is $\frac{3}{16}$ " and that the piece is to be made of drill rod. His drill and reamer size must be determined from the stock size of the pin and he must also take into consideration that on standard stock there is always a commercial variation.

Not being thoroughly familiar with the accuracy of this job, especially if he is an inexperienced tool maker, he must find out whether or not the taper machined on the end of part 3 is eccentric with the thread. Referring to the blueprint, he finds to the left of the front elevation that the point must run true with the thread.

QUIZ QUESTIONS*

1. What does the specification—Make one .8125"—indicate?
2. The head of the bushing is shown to be knurled. Are any specifications given for this, and what would the machinist do?
3. On Detail No. 22, what does the invisible line in the front elevation suggest to the tool maker?
4. In making tapered pins, Detail No. 3, where does the tool maker find out what material to use?

***No written answers are required for Quiz Questions.**

In punch and die drafting the same practices apply as in jig and fixture drafting. Some plants prefer detailing each piece part, whereas, others specify all the dimensions, notes, etc., on the assembly drawing.

Punch and Die. A typical drawing of a punch and die is shown in Fig. 70. One view of the complete assembly is shown in the front sectional elevation. In the plan view the die, stripper, etc., are shown. At the right of this drawing is a plan view of the punch holder.

From a study of the plan view of the die, we find that a shoe is indicated by Detail No. 1. On this shoe a stripper is fastened with hexagonal headed screws, indicated by Detail

No. 5. Detail No. 12 shows a dowel pin and Detail No. 2 shows a gauge block or stop.

In the front sectional elevation, we find that Detail No 7 is a $\frac{3}{4}$ " punch, and that there are five such punches. Below the punches, button dies can be seen. The one near the center is Detail No. 9.

At the left end of the die shoe, we find a gauge block which is fastened with a screw. This block is prevented from moving by using two dowel pins, as indicated by Detail No. 4.

Let us assume that the die maker is ready to start on his job. On his drawings, he finds no dimensions given for the die shoe other than $2\frac{1}{2}$ ", which is the height; therefore, he knows that this die shoe is standard and is one that can be procured from the stockroom.

Before making the button dies, the die maker studies the front sectional elevation. To the right of the center line near a button die, he finds the diameter is to be $1\frac{1}{4}$ " and the over-all height is to be $1\frac{1}{2}$ ". As the other dimensions needed are standard practice, they are not given on the blueprint.

Before making the punches, the die maker again studies the blueprint. He finds that three of the punches are to be $3\frac{1}{8}$ " long, and two of them $3\frac{5}{8}$ " long. He also finds that all the punches are to be inserted in the punch holder to a depth of $1\frac{1}{2}$ ". As the diameter and the angle for the lower end of the punches are not given, he follows standard practice. In some cases the stock size for such punches is indicated in the bill of material. So he refers to the bill of material and finds the stock size for the punches is $\frac{3}{4}$ ". He also finds that the sizes of the hexagonal headed screws, Detail No. 5, are to be $\frac{1}{2}$ " diameter by 2" long and that six of these screws are required and that they are standard.

In providing for the locations of the button dies when laying out the shoe, the die maker must find out how far apart these dies are to be spaced. On the plan view, he finds that the second die is to be 2.500" from the first die, the third one 4.000" from the second, the fourth 2.000" from the third, and the fifth 6.250" from the fourth die. Now he looks for the spacing for the dowel pins that pass through the stripper plate. In the plan view, he finds that one pin is spaced at a distance of 6" to the left of the perpendicular center line and the other 6" to the right of the perpendicular center line.

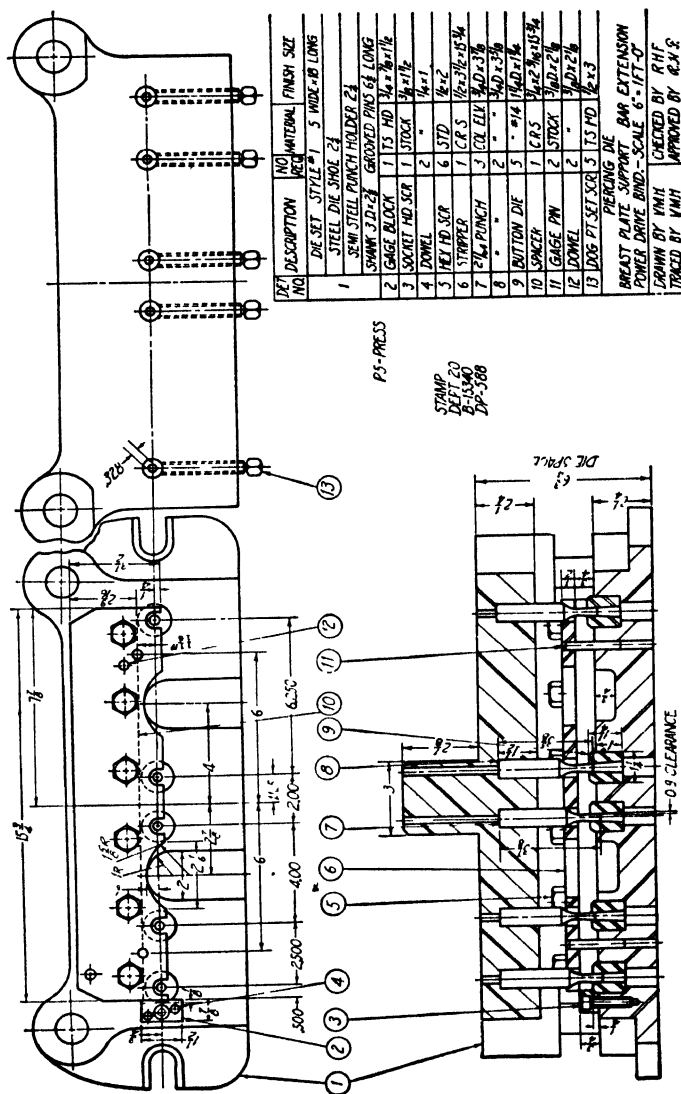


Fig 70. Working Drawing for a Piercing Die

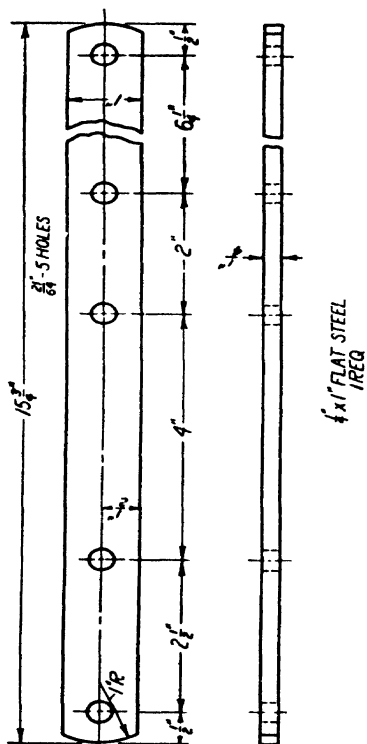


Fig. 71. Working Drawing for a Breast Plate

In the plan view below the stripper plate and very nearly tangent to the button dies is an invisible horizontal line. This horizontal line is used to designate the side of the stripper plate, which is used as a guide. Now the die maker must find out the distance from the back of the stripper plate to the front side used as a guide. He studies the plan view of the dies, shoes, etc. At the right end of the stripper plate, he finds the distance is to be $2\frac{9}{16}$ ". On the right end of the stripper plate, he finds the distance from the guide on the stripper plate to the center line of the button dies is to be .500".

The size of the gauge block, which is really a stop, must now be found. The mechanic looks at the plan view and, at the left end of the stripper plate, he finds that the gauge block, Detail No. 2, is to be $1\frac{1}{2}$ " long and $\frac{7}{8}$ " wide. The height, which he finds on the front sectional elevation, is to be $\frac{3}{4}$ ".

The button dies project a certain distance above the shoe. Referring to the front sectional elevation at the left end of it, he finds that this distance is to be $\frac{1}{4}$ ".

Now the mechanic wants to know how the punches are prevented from working loose. In the plan view of the punch holder, he finds that five set screws, Detail No. 13, are used.

The die and punch shown in blueprint, Fig. 70, is a piercing die for a breast plate supporting a bar extension power drive binder. Fig. 71 shows a blueprint of the breast plate, etc., and from a study of it the student will understand that the die and punch, Fig. 70, is used for piercing the five $\frac{3}{4}$ " holes.

Comparing the center distances of these holes, Fig. 71, it will be seen that the distance for the pair of holes at the left end in the bar is $2\frac{1}{2}$ ", which is equivalent to the 2.500", as indicated in Fig. 70. The center distance for the next pair of holes is 4", the same as shown in Fig. 71.

The $\frac{1}{4}$ " x 1" flat steel material used for the breast plate is placed on top of the five button dies when it is desired to pierce the holes.

QUIZ QUESTIONS*

1. What are the sizes of the five punches which are used?
2. What is the stock size used for making these punches?
3. How far apart are the dies to be spaced?

4. What is the spacing for the dowel pins that pass through the stripper plate?
5. What is the distance from the back of the stripper plate to the front side used as a guide?
6. What is the size of the gauge block?

***No written answers are required for Quiz Questions.**

INSTALLATION DRAWINGS

The millwright is the man who is particularly concerned about installation drawings, but many a mechanic is often called upon to do work of this kind.

Fig. 72 is a comparatively simple blueprint of a motor, a line shaft, a countershaft for a special boring machine, and a countershaft for an engine lathe. It can be assumed that the two walls are part of a factory building. Therefore, let us consider Fig. 72 as being a corner of a machine shop in which various machines are to be installed.

On some blueprints, like the one shown in Fig. 72, the belts are not shown as a competent mechanic does not really need this information. As the directions of rotation are shown by arrows near the pulleys, the millwright is able to determine the types of belts to run over a certain pair of pulleys.

Another feature about the drawing, Fig. 72, is that the outlines of the machines themselves are not shown on the blueprint, only the center lines are given. These show the position of the machines, which is sufficient for a competent millwright.

The blueprint shows the mechanic that three shaft hangers and the line shaft are in place and the first hanger is 1' 10" from the wall.

Let us see just what the millwright is concerned about on this job. He must first find out where the electric motor is to be located. He looks at the blueprint and finds that this motor is to be located 4'-3" from the wall which is at right angles to the motor shaft. Now he looks for the distance from the left wall to the center line of the motor shaft. He finds at the top of the drawing to the right side of the left wall that the distance is to be 20". And he finds the distance from the center of the motor shaft to the center of the line shaft is to be 15'-0".

The mechanic is now ready to install the countershaft for the boring machine. He must know just where it is to be located. From his blueprint, he finds that the first hanger is to be located at a distance of 4'10" from the wall and the countershaft is to be 10'0" from the line shaft.

To install the countershaft for the engine lathe, the millwright must know what the distance is to be from the wall to the center of the shaft hanger, also the distance the countershaft is to be located from the line shaft. From his blueprint, he finds that the shaft hanger is to be located at a distance of 7'4" from the wall and that the countershaft is to be 10'0" to the right of the line shaft.

The letters, *E*, *H*, *G*, and *F* indicate to the mechanic just where certain pulleys are to be located so there is no possibility of confusion. If this was not done, there would be a chance for the pulleys *H* being located where the pulleys *F* and *G* should be. If pulleys *F* and *G* were located where pulleys *H* should be, then the speed of the countershaft for the engine lathe would be affected. The draftsman in showing the designation mark must be very careful that the pulleys are marked correctly to avoid trouble.

QUIZ QUESTIONS*

1. What are the dimensions for locating the motor?
2. How far from the wall is the line shaft?
3. How is the countershaft for the engine lathe located?
4. How is the direction of the turning of the shafts indicated?
5. What is the speed of the motor?
6. What is the speed of the line shaft?

*No written answers are required for Quiz Questions.

Centrifugal Pump. Fig. 73 is a blueprint of a centrifugal pump. Two views are shown—a front elevation and a right side elevation. The drawings may be considered as a combination installation and general assembly drawing.

In installing the pump, it is necessary to know how many bolts are required for making the connection on the suction end of the pump. This information is found on the front elevation in the specification—8 holes $\frac{1}{8}$ " drill. This shows that 8 holes are required. If it is necessary to drill the holes in the

connecting piece before the pump is received, a blueprint could be mailed to the customer which would show him that the 8 holes were to be equally spaced and drilled with a $\frac{1}{8}$ " drill. He would also need to know the bolt circle diameter. At the left of the front elevation he finds this diameter is to be $11\frac{1}{4}$ ". Now he must find out what the size of the waste pipe connection is to be. The specification calls for an 8" waste pipe connection. The size of the priming connection is also another important consideration. At the right of the pump he finds the specification— $3\frac{1}{2}$ " priming connection.

The customer might also want to know the diameter and thickness of the flange on the discharge end. On the right side elevation, he finds that the diameter is to be $13\frac{1}{2}$ " and that the thickness is to be $1\frac{1}{4}$ ".

The customer now asks, "Will the pump, motor, etc., fit into the space planned on?" He looks at his blueprint for the over-all length and finds it is to be $90\frac{1}{4}$ " and the width is to be $57\frac{1}{4}$ ". He compares these dimensions with those of the space planned on.

In fitting up the motor, pump, coupling, etc., for installation, it is necessary for the customer to know the distance from the center of the pump to the right side of the coupling that is attached to the pump shaft. On the right side elevation and above it, he finds this distance is to be $24\frac{1}{2}$ ". To find the distance from the center of the motor to the side of the coupling member that is attached to the motor shaft, he looks above the motor and finds the specification $25\frac{7}{16}$ ".

QUIZ QUESTIONS*

1. What is the distance from the center of the pump to the center of the suction end?
2. What is the size of the priming connection?
3. What is the diameter of the bolt circle at suction end of pump?
4. What is the size of opening at discharge end?
5. What is the distance from base of pump to center of discharge opening?
6. What size of pipe tap should be used for hose valve connection?

*No written answers are required for Quiz Questions.

PRACTICAL MATHEMATICS

PART I

INTRODUCTION

No one who is at all acquainted with the demands of the Engineering profession will deny the need of a good foundation in elementary mathematics any more than he will deny the need of a solid underpinning on which to rest the walls of a big business block.

The simplest problems of the contractor and workman, such as the number of feet of lumber required for a house, the number of cubic yards of excavation for a ditch or cellar, the proper understanding of plans and specifications, and the laying off of measurements according to these plans, all require a knowledge of this important subject. The size of a concrete retaining wall, the dimensions of a girder for a steel structure, the amount of iron in the field of a dynamo, or the capacity of the cylinders of an engine, is certainly not left to the arbitrary judgment of a foreman but is carefully worked out by mathematics and by a knowledge of the properties of the materials used.

Mathematics might be likened to a kit of tools which the workman carries; the master workman carries more than the apprentice and the more tools each man has in his kit and knows how to use, the more things he can do and the greater is his earning power. Each mathematical process is a tool to be used as the occasion demands. Some of them are used in every problem which comes up, others less frequently, but the more advanced the work the greater the number of tools required.

It is with this keen demand in mind, therefore, that we are requiring of each student at the outset of his course this work or its equivalent in Practical Mathematics. We want him to fill his kit with enough tools to meet the steady demands of the work ahead of him, and we feel sure that, once provided with this equipment, his progress will be assured.

In the preparation of this work the authors have intentionally lost sight of the material usually found in the school books on this subject, and have kept in mind only the particular parts which are of special importance to the engineering student. Not only the topics discussed but all of the problems have been made exceptionally practical, and the aim has been at all times to give the student the satisfaction of knowing that whatever he is learning will be of use in his work and will also count for his advancement.

DEFINITIONS AND MATHEMATICAL SIGNS

1. Definitions. *Mathematics* is the science which treats of quantity; its fundamental branches are Arithmetic, Algebra, and Geometry.

Quantity is anything which can be increased, diminished, or measured; as numbers, lines, space, motion, time, volume, and weight.

A *unit* is a single thing, or *one*.

A *number*, or quantity, is a unit or a collection of units and is either *concrete* or *abstract*.

A *concrete* number, or quantity, is one whose units refer to particular things, as, for example, 5 rivets, 7 bolts.

An *abstract* number, or quantity, does not refer to any particular thing. For example, 5, 23, etc., used without designating any particular objects, are abstract numbers.

Like quantities refer to the same thing, as 7 saws, 2 saws; *unlike* quantities refer to different things, as 2 trunks, 4 tables, 3 chairs.

2. Mathematical Signs. For the sake of brevity, signs are used in mathematics to indicate processes. Those signs most used in Arithmetic are $+$, $-$, $\times$, $\div$, $=$, $()$, and $\frac{\quad}{\quad}$.

The sign $+$ is read "plus" and is the sign of addition. It shows that the quantities between which it is placed are to be added together. If 2 and 2 are to be added, it is expressed, thus: $2+2$ are four.

The sign $-$ is read "minus" and is the sign of subtraction. It means that the quantity which follows this sign is to be subtracted or taken away from the quantity which precedes it, thus: $6-4$ are 2.

The sign $\times$ is read "times" and is the sign of multiplication. It means that the quantity which precedes this sign is to be multiplied by the quantity which follows it, thus: 2×5 are 10.

The sign $\div$ is read "divided by" and is the sign of division. It means that the quantity which precedes this sign is to be divided by the quantity which follows it, thus: $4 \div 2$ are 2.

The sign $=$ is read "equals" or "is equal to" and is the sign of equality. It means that the expressions between which it is placed are identical in value, thus: $4 + 3 = 10 - 3$. This sign is very often misused. Great care should be taken at all times to make sure that the quantities connected by it are *equal*. For example, it would be absurd to say that $5 + 9 = 14 \div 2 = 7$, because $5 + 9$ does not equal 7.

The *parenthesis* () and *vinculum* — are used to show that two or more quantities are to be treated as one; or in other words, that the operations indicated within the parenthesis or under the vinculum are to be carried out first, thus: $-(20+5)$ is equal to -25 , not $-20+5$; similarly, $-2+3$ is equal to -5 .

NOTATION

3. *Notation* is the art of writing numbers in words, in figures, and in letters.

There are two methods of notation in common use; the *Roman* and the *Arabic*.

4. **Roman Notation.** In the *Roman notation*, 7 capital letters are used, as follows:

I	is used to express one.
V " " " "	five.
X " " " "	ten.
L " " " "	fifty.
C " " " "	one hundred.
D " " " "	five hundred.
M " " " "	one thousand.

All other numbers are expressed by repetitions or by combinations of these seven letters according to the following rules:

By repeating a letter the value denoted by the letter is doubled; thus: XX means twenty; CC means two hundred.

By placing a letter denoting a less value before a letter denoting a greater, their difference of value is represented; thus: IV denotes four or one less than five; XL denotes forty or ten less than fifty.

By placing a letter denoting a less value after a letter denoting a greater value, their sum is represented; thus: VII denotes seven or two more than five. XV denotes fifteen or five more than ten.

A line — placed over a letter increases the value denoted by the letter a thousand times; thus: $\overline{X}$ means ten thousand. $\overline{IV}$ means four thousand.

The use of the Roman notation is now confined mainly to the writing of dates, and the numbering of chapters in books, and the hours on the dials of clocks.

Table I shows some of the combinations of the 7 letters used.

TABLE I
Roman Notation

I.....one	LXXX.....eighty
II.....two	XC.....ninety
III.....three	C.....one hundred
IV.....four	CC.....two hundred
V.....five	CCC.....three hundred
VI.....six	D.....five hundred
VII.....seven	DC.....six hundred
VIII.....eight	DCC.....seven hundred
IX.....nine	CM.....nine hundred
X.....ten	M.....one thousand
XX.....twenty	MID.....fifteen hundred
XXX.....thirty	MM.....two thousand
XL.....forty	X.....ten thousand
L.....fifty	$\overline{M}$ one million
LX.....sixty	MCMX.....1910
LXX.....seventy	

5. Arabic Notation. The Arabic notation employs ten characters or figures in expressing numbers. They are

1, 2, 3, 4, 5, 6, 7, 8, 9, 0
one two three four five six seven eight nine cipher

The first nine are sometimes called *digits*; the cipher is also called *naught* or *zero* because it expresses *nothing* or the absence of a number.

The digits (1, 2, 3, 4, 5, 6, 7, 8, 9) have been termed *significant figures* because each has of itself a definite value, always representing so many units or *ones* as its name indicates. However, the value of the units represented by a figure depends upon the particular position which that figure occupies with regard to other figures. This position is called its *place* or *order*.

For example, if three figures are written together to represent a number, as 444, each of these figures, without regard to its place, expresses four units, but when considered as part of the number these fours differ in value. The 4 in the first place to the right represents 4 units; the 4 in the second place, represents 4 tens or 4 units each ten times the size or value of a unit of the first place; and the 4 in the third place, represents 4 hundreds, or 4 units each one hundred times the size or value of a unit of the first place. It is readily seen that the value of any figure is increased ten-fold by removing it one place to the left.

The cipher becomes significant when connected with other figures by filling a place that otherwise would be vacant, as in 10 (ten) it gives a ten-fold value to the 1. In 130 (one hundred thirty) it gives a ten-fold value to the 13. A cipher between two or more figures produces the same effect. In 405 the cipher which fills the intervening place between 4 and 5 causes the 4 to represent four hundreds, not four tens.

The following principles should be firmly fixed in mind:

All numbers are expressed by the nine digits and zero.

Zero has no value; it is used to fill vacant places only.

A figure has different values according to the place it occupies.

The base of the system of notation is ten; ten units of any order making one unit of the next higher order.

NUMERATION

6. *Numeration* is the art of reading numbers when expressed by letters or figures.

This is accomplished by first enumerating the *orders* from right to left, as shown in Table II, and then reading these orders in the reverse direction in groups of three, called *periods*. The first three orders, *Units, Tens, Hundreds*, constitute the *first* or *unit period*. The second three orders form the *second* or *thousand period*; the third three orders, the *third* or *million period*; and so on.*

* Commas are used to separate the different periods of figures. In reading numbers never use "and" to connect the periods, or orders.

The system of periods is shown clearly in Table III. Such a number as 534 consists of one period and is read—five hundred thirty-four. The number shown below the orders in Table II is read—two million, seven hundred fifteen thousand, six hundred thirty-nine.

TABLE II
The Orders

7th	6th	5th	4th	3rd	2nd	1st
2 Millions	7 Hundreds of Thousands	1 Tens of Thousands	5 Thousands	6 Hundreds	3 Tens	9 Units

The number in Table III, which is 987,654,321,987,654, is divided into the periods 987, 654, 321, 987, 654, and is read—nine hundred eighty-seven trillion, six hundred fifty-four billion, three hundred twenty-one million, nine hundred eighty-seven thousand, six hundred fifty-four.

TABLE III
The Periods

Hundreds Tens Units	Hundreds Tens Units	Hundreds Tens Units	Hundreds Tens Units	Hundreds Tens Units
987	654	321	987	654
5th Period	4th Period	3rd Period	2nd Period	1st Period

PROBLEMS FOR PRACTICE

Write in words:

1. 18,765,972.
2. 834,769,780.
3. 3,576,879,421.
4. 10,805,056. Ans. Ten million, eight hundred five thousand, fifty-six.

Write in figures:

5. Seventy-eight million, forty-one thousand, seven.
6. One thousand three. Ans. 1,003.
7. Five hundred six thousand.
8. Ninety million, two thousand, three hundred twenty-seven
9. Three hundred five thousand, seventy-nine.
10. Eight hundred sixty-four million, four thousand, twenty.

ADDITION

7. *Adding* is the process of finding a number which is equal to the combined values of two or more given numbers. The result thus obtained is called the *sum*. Hence, it may be said that the sum of two or more numbers is a number containing as many units as all the numbers taken together. Thus the sum of 5 rivets and 7 rivets is 12 rivets, since 12 contains as many units as 5 and 7 together.

Letters of the alphabet are also used to represent quantities. A letter may stand for any number, but the same letter must have one value throughout a given problem, although it may have different values in different problems. For example, a may equal 3 in one problem, and 7 or 13 in another; b may equal 2 in one problem, and 5 in another; c may equal 2, 3, 9 or 25; and so on with other letters. The sum of a , b , and c will be equal to $a+b+c$, just as the sum of 2, 5, and 7 will be equal to $2+5+7$. Now, if $a=2$, $b=5$, and $c=7$, the sum of a , b , and c is found by simply adding together 2, 5, and 7, obtaining 14; on the other hand, if the values of a , b , and c are not known, their sum can only be indicated. If a and b represent the horse-powers of two different engines or of two weights lying in a scale-pan, then $a+b$ equals the total horse-power or the total weight.

8. Since a number is a collection of units of the same kind, two or more numbers may be united into one sum only when they are like quantities; thus: 2 bolts + 4 bolts = 6 bolts, but 4 rivets and 3 bolts cannot be united into one quantity, either of bolts or of rivets. On the other hand, if quantities of different kinds are represented by letters, they may be added together in order to help solve the problem. For example, if a represents a car, and b an engine, their sum may be indicated ($a + b$). It must be remembered, however, that the car and the engine have not actually been added together, because they represent unlike quantities. The use of letters makes the solution of certain kinds of problems much simpler, and later will be taken up more fully.

Rules for Addition. *Write the numbers under each other, placing them so that units are under units, tens under tens, hundreds under hundreds, and so on.*

Add up the column of units; and put the right-hand figure of this sum under the unit column, carrying the remaining figure or figures to the column of tens; add up the tens column, including the carried figures, put down the right-hand figure and carry as before. Continue in this way until the last column is reached, putting down the total of the last column to give the final sum.

Examples. 1. Find the sum of 567; 141; and 93.

SOLUTION. Write these numbers under one another, so that the units of each shall be in the same vertical column. Then add up as follows: 3 and 1 are 4, and 7 are 11. Place the right-hand figure 1 under the units column and carry 1 ten to the next column. Adding the tens column, 1 (carried) + 9 are 10 and 4 are 14 and 6 are 20. Put down the right-hand figure, which is zero, and carry 2 to the next column; then 2 (carried) and 1 are 3 and 5 are 8. Putting down the 8, the total sum is found to be 801.

567
141
93
801

2. Find the sum of 6,321; 2,576; 9,702; and 257.

SOLUTION. Write these numbers one under the other as before. Add up the unit column as follows: 7 and 2 are 9, and 6 are 15, and 1 are 16. Put down the 6 and carry the 1; then adding the tens column, 1 (carried) and 5 are 6, and 7 are 13, and 2 are 15. Put down the 5 and carry the 1; then adding the hundreds column, 1 (carried) and 2 are 3, and 7 are 10, and 5 are 15, and 3 are 18; put down the 8 and carry the 1. Then adding the thousands column, 1 (carried) and 9 are 10, and 2 are 12, and 6 are 18. Putting down the whole amount, 18, gives the total sum of the numbers as 18,856.

6321
2576
9702
257
18856

3. Add the following quantities: $a + 2b + c$, $3b + 2c$, $6a + b + c$, and $10b$.

SOLUTION. Write the quantities in columns, keeping the like terms under each other; that is, keep the a 's in the first vertical column, the b 's in the second, the c 's in the third. The sum of the a 's is $a + 6a$ or $7a$.^{*} $2b + 3b + b + 10b$ are $16b$. $c + 2c + c$ are $4c$. Connecting the sums by the sign of addition the total is found to be $7a + 16b + 4c$. When the number showing how many times the letter is taken is omitted, the letter is to be taken once. Thus: $a = 1a$, $b = 1b$, $c = 1c$.

$$\begin{array}{r} a + 2b + c \\ 3b + 2c \\ 6a + b + c \\ 10b \\ \hline 7a + 16b + 4c \end{array}$$

9. **Proof.** To prove that a sum is correct, begin at the top and add the columns *downward* in the same manner as they were added upward; if the two sums agree, the work is presumably correct, for adding downward inverts the order of the figures, and therefore any error made in the first addition would probably be detected in the second.

PROBLEMS FOR PRACTICE

Find the sum of

1. $56 + 49 + 17 + 36 + 21$. Ans. 179
2. $42 + 46 + 43 + 58 + 91$.
3. $467 + 536 + 84 + 705$. Ans. 1,792
4. $2,008 + 1,400 + 706 + 300 + 77$.
5. $8,950 + 15,765 + 7,732$. Ans. 32,447
6. $26,661 + 8,735 + 6,877 + 33,413$.
7. $8,792 + 980 + 5,607 + 89$.
8. $346 + 4,682 + 64 + 798 + 21$.
9. $26 + 425,902 + 3,006 + 490 + 36,221$. Ans. 465,645
10. $3a + 11b + a + 7a + 5b$. Ans. $11a + 16b$
11. $c + 2a + 5c + 4a + 3c$.
12. $6a + b + 5d + 2d$. Ans. $6a + b + 7d$
13. $4a + 2b + 3a + 7c + 3c$.

10. It has now been shown how to add numbers correctly when the process is indicated. It often happens, however, that problems will be met in which only the statement of the relations between quantities are given. The following illustrative example will show the method of solving problems of this nature.

^{*}7 is called the *coefficient* of a and indicates that a is to be taken seven times. See §14, p. 15.

Example. An electric light plant, capable of furnishing current for 200 16-candle-power lamps cost as follows:

16 Horse-power engine.	350	dollars
Dynamo.	275	"
Driving belt.	50	"
Installation.	35	"

What was the total cost?

SOLUTION. In an example of this kind the problem should	350
be read several times until it is thoroughly understood. If the	275
numbers to be used in computation are fixed in mind the student	50
will not be misled by the wording. For example, it can readily be	35
seen that 200 will not be used in the computation. Find the total	- 710
cost of the equipment as shown in the margin.	

PROBLEMS FOR PRACTICE

1. A marine engine during a 3 hours' run makes 9,187 revolutions the first hour, 9,062 the second, and 9,233 the third. How many does it make in the 3 hours?

2. Coal is fed to a furnace as follows: Monday, 376 pounds; Tuesday, 307 pounds; Wednesday, 438 pounds; Thursday, 425 pounds; Friday, 399 pounds; Saturday, 301 pounds. Find the total for the week.

3. The items for lumber called for in a contract were—frame, 3,896 feet; flooring, 6,796 feet; finish, 2,739 feet. How many feet were used?

4. A surveying party works six weeks. The first week they survey 151 miles; the second week, 111 miles; the third week, 162 miles; the fourth week, 159 miles; the fifth week, 96 miles; the sixth week, 48 miles. How many miles did they survey?

5. There are five water wheels installed in a water power plant. The power furnished by the first wheel is 2,225 horse-power, and the others furnish, 3,150, 4,275, 5,650, and 8,275 horse-power. What is the total capacity of the five wheels?

6. The weekly capacity of 4 lathes is as follows: 2,500 castings, 4,175 nuts, 3,420 brass boxings, and 2,185 finished trimmings. How many pieces do the four lathes turn out per week?

7. When purchasing an 85 dollar indicator, the following extras were bought:

1 Spring.....	5 dollars
1 Elbow.....	3 "
Pantograph.....	10 "
Speed Counter.....	2 "
Planimeter.....	20 "

What was the total cost of the outfit including the indicator?

8. If a 10-inch belt will transmit 17 horse-power at a speed of 1,800 feet per minute, and a 16-inch belt will transmit 36 horse-power at the same speed, how much power will be transmitted by the two belts?

SUBTRACTION

11. *Subtraction* is the process of finding the difference between two quantities; this difference when added to the smaller will give a result equal to the greater.

For example, the difference between 16 and 7 is 9, since 7 added to 9 makes 16. The greater of the two quantities whose difference is to be found, is called the *minuend*; the smaller is called the *subtrahend*. The quantity left after taking the subtrahend from the minuend is called the *difference* or *remainder*.

Only like quantities and units of the same order can be subtracted, and the remainder is always like the minuend and subtrahend.

It can readily be seen that subtraction is the reverse of addition, and this fact is made use of to prove subtraction as shown in the following:

Example. Subtract 114 from 237 and prove the result.

SOLUTION. Beginning with the units column, 4 (units) are subtracted from 7 (units) leaving 3 (units), which is set down directly under the column in units place. Proceeding to the next column 1 (ten) is subtracted from 3 (tens) leaving 2 (tens), which is set down in tens place. Proceeding as before, the final remainder is found to be 123.

	Proof
237	114
<u>114</u>	<u>123</u>
123	237

12. **Rules for Subtraction.** Write the subtrahend under the minuend so that units of the same order will be in the same column.

Begin with the units of the lowest order to subtract, and proceed to the highest, writing each remainder under the line in its proper place.

If any digit of the minuend is less than the corresponding digit of the subtrahend, add ten to it and then subtract; but consider that the next digit of the minuend has been diminished by one.

Examples. 1. From 6,784 subtract 3,776.

SOLUTION. In order to subtract 6 in the units place in the subtrahend from 4 in the units place in the minuend, the 4 must first be increased by 10 giving 14; then 6 subtracted from 14 leaves 8 in the units place of the remainder. But the 10 added to the 4, has been obtained by diminishing the 8 in the tens place of the minuend by 1, leaving 7. The 7 in the tens place of the subtrahend subtracted from this leaves zero in the tens place of the remainder. The 7 in the hundreds place of the subtrahend subtracted from the 7 in the hundreds place of the minuend leaves zero in the hundreds place of the remainder, and the 3 in the thousands place of the subtrahend subtracted from the 6 in the thousands place of the minuend leaves 3 in the thousands place of the remainder, giving 3,008.

6784 minuend
3776 subtrahend
3008 remainder

2. From 1,900 subtract 621.

SOLUTION. The 0 in the units place of the minuend must first be increased by 10; then 1 subtracted from 10 leaves 9 in the units place of the remainder. In adding 10 to the 0, 1 has been taken from the tens place, but as it was itself 0, it had to borrow from the next place and continue borrowing until a numerical place was reached. Proceeding in this way the total remainder 379 is obtained.

1000 minuend
621 subtrahend
379 remainder

13. Letters may be used in subtraction just as in addition. For instance, if a quantity represented by b is to be subtracted from a quantity represented by a , the difference will be represented by the quantity $(a-b)$; or if c is to be subtracted from b , the difference will be represented by the quantity $(b-c)$. If the quantity consists of a letter or letters with a coefficient, as $8ab$, and another term containing the same letters, as $3ab$, is to be subtracted from it, the subtraction may be performed in the usual way. It must be remembered, however, that $8ab$ means that ab is taken eight times, while $3ab$ means that ab is to be taken three times. Therefore, if $3ab$ is taken from $8ab$, $5ab$ remains. In other words, $8ab-3ab$ equals $5ab$.

Examples. 1. From $7ab + 9b + 4c$ subtract $3ab + b + 2c$.

SOLUTION. Arrange the terms as in addition of letters, subtracting the coefficients only and bringing down the letter or letters after each result. Thus: $7ab - 3ab = 4ab$; $9b - b = 9b - 1b = 8b$; $4c - 2c = 2c$. The separate results of the subtraction may now be connected by the plus sign, giving as the final answer the quantity $4ab + 8b + 2c$.

$$\begin{array}{r} 7ab + 9b + 4c \\ 3ab + b + 2c \\ \hline 4ab + 8b + 2c \end{array}$$

2. From $6a + 2d$ subtract $5a + 2d$.

SOLUTION. Subtracting we have $6a - 5a = 1a = a$; $2d - 2d = 0d = 0$. When the difference between the coefficients is 0, the term is 0. Therefore, the second term of the remainder disappears, giving as the final result the quantity a .

$$\begin{array}{r} 6a + 2d \\ 5a + 2d \\ \hline a \end{array}$$

PROBLEMS FOR PRACTICE

1. From 7,282 subtract 4,815 Ans. 2,467
2. From 64,037 subtract 5,908.
3. From 6,231 subtract 3,084. Ans. 3,147
4. From 1,740,932 subtract 807,605.
5. From 71,287 subtract 40,089. Ans. 31,198
6. From 1,000,000 subtract 999,999.
7. From the sum of 2,465,321 and 975,803
subtract 739,034. Ans. 2,702,090
8. From $9b + 7c$ subtract $4b + c$.
9. From $12a + 15b + 6c$ subtract $8a + 14b + 6c$.
10. From $5a + 2b$ subtract $3a + b$. Ans. $2a + b$
11. From $a + 5b + 2d$ subtract $a + 4b + 2d$. Ans. b
12. From a tank containing 935 gallons of water, 648 gallons were drawn off. Then 247 gallons ran in. How many gallons were then in the tank? (Suggestion: Subtract 648 from 935 and add 247.)
13. A man purchased 8,983 bricks, but used only 5,363. How many had he left?
14. A coal shed contains 8,579 tons. 3,243 tons are taken from it. It then receives 4,112 tons more. After that 1,602 tons are taken out of it. How many tons remain?
15. An electric power plant can generate 2,000 horse-power. Of this, 1,910 horse-power is used. The manager then agrees to

furnish another firm with 784 horse-power. How much more power will he need? (Suggestion: Add 784 to 1,910 and subtract 2,000.)

Ans. 694 horse-power.

16. An engine develops 147 horse-power. 16 horse-power is used in running the engine itself. How much power is available for running machinery?

17. 1,200 gallons are pumped from a tank. Of this, 32 gallons are lost in leakage, etc. How much is discharged by the pump?

18. A 75 horse-power boiler evaporates 2,140 pounds of water into steam per hour. One engine uses 1,310 pounds, another uses 417 pounds, and a pump requires the remainder. How much steam is used by the pump? (Suggestion: Add 417 to 1,310 and subtract from 2,140.)

Ans. 413 pounds.

19. An engine makes 54,000 revolutions in a day of 12 hours. A motor makes 720,000 revolutions in the same time. By how many revolutions per day does the speed of the motor exceed that of the engine?

20. It takes 3,880 pounds of steam per hour to run a certain 160 horse-power engine. If it takes 1,940 pounds to run a 60 horse-power engine and 2,140 pounds to run a 72 horse-power engine, does the largest engine require as much steam as the two small ones? (Suggestion: Add 1,940 to 2,140 and compare with 3,880.)

MULTIPLICATION

14. *Multiplication* is a short method of adding a quantity to itself a certain number of times; or, it is the process of taking one quantity as many times as there are units in another.

It is known that $2 + 2 + 2 + 2 + 2 = 10$; but this same process may be expressed more briefly by the aid of multiplication, thus: $5 \times 2 = 10$. The 5 shows how many twos are used in adding. This last expression is read, "five times two equals ten."

In multiplication three terms are employed—the multiplicand, the multiplier, and the product.

The *multiplicand* is the quantity to be multiplied or taken.

The *multiplier* denotes the number of times the multiplicand is to be taken.

The *product* is the result or quantity obtained by the multiplication.

Another term, called the *coefficient*, is used to indicate the numerical part of a quantity which consists of numbers and letters multiplied together. Thus in the expression $3ab$, 3 is the coefficient of ab , and denotes that ab is taken 3 times. Sometimes it is convenient to consider any part of the product as the coefficient of the remaining part. Thus in $3ab$, $3a$ might be considered the coefficient of b , or $3b$ the coefficient of a . A coefficient is, therefore, called *numerical* or *literal* according as it is a number or one or more letters. When no numerical coefficient is expressed, 1 is always understood,

TABLE IV
Multiplication Table

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50
3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72	75
4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80	84	88	92	96	100
5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	105	110	115	120	125
6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120	126	132	138	144	150
7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133	140	147	154	161	168	175
8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152	160	168	176	184	192	200
9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162	171	180	189	198	207	216	225
10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200	210	220	230	240	250
11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198	209	220	231	242	253	264	275
12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240	252	264	276	288	300
13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247	260	273	286	299	312	325
14	28	42	56	70	84	98	112	126	140	154	168	182	196	210	224	238	252	266	280	294	308	322	336	350
15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285	300	315	330	345	360	375
16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	272	288	304	320	336	352	368	384	400
17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	289	306	323	340	357	374	391	408	425
18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288	306	324	342	360	378	396	414	432	450
19	38	57	76	95	114	133	152	171	190	209	228	247	266	285	304	323	342	361	380	399	418	437	456	475
20	40	60	80	100	120	140	160	180	200	220	240	260	280	300	320	340	360	380	400	420	440	460	480	500
21	42	63	84	105	126	147	168	189	210	231	252	273	294	315	336	357	378	399	420	441	462	483	504	525
22	44	66	88	110	132	154	176	198	220	242	264	286	308	330	352	374	396	418	440	462	484	506	528	550
23	46	69	92	115	138	161	184	207	230	253	276	299	322	345	368	391	414	437	460	483	506	529	552	575
24	48	72	96	120	144	168	192	216	240	264	288	312	336	360	384	408	432	456	480	504	528	552	576	600
25	50	75	100	125	150	175	200	225	250	275	300	325	350	375	400	425	450	475	500	525	550	575	600	625
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

as a is the same as $1a$. When a coefficient occurs just before a parenthesis, it indicates that every term within the parenthesis is to be multiplied by that coefficient.

To multiply with accuracy and rapidity, the product of any two quantities, at least from 2 to 12, must be known at sight. The combinations of these should be practiced until they can be given correctly and without hesitation.

The following points should also be fixed firmly in mind: Zero times any quantity or any quantity times zero is zero. For example,

$0 \times 0 = 0$; $0 \times 8 = 0$; $942 \times 0 = 0$, $b \times 0 = 0$, etc. One times any quantity or any quantity times one gives that quantity as a product. For example, $1 \times 7 = 7$, $85 \times 1 = 85$, $1 \times 1 = 1$, $a \times 1 = a$.

Table IV gives the product of any two numbers up to 25×25 . To use this table, find one of the two numbers in the upper row and the other in the left-hand column. The product of the two will be found at the intersection of two imaginary lines drawn parallel to the heavy lines shown in the table at 12×12 and 20×20 . For example under the 9 and opposite the 8 is found the product, 72. That is, $9 \times 8 = 72$.

15. Rules for Multiplication. (a) *In performing multiplication, treat both terms as abstract quantities, always using the larger quantity as the multiplicand. After the result is obtained, determine from the nature of the problem in what units the result should be expressed.*

(b) *Place right-hand digit of multiplier directly under right-hand digit of multiplicand (with one exception—See Case 2, p. 17).*

(c) *Each figure of the multiplicand is multiplied by each significant figure of the multiplier, and the right-hand figure of each product is placed under the figure of the multiplier used to obtain it. The sum of the several products will be the entire product. When there is a zero in the multiplier, multiply by the significant figures only, taking care to place the right-hand figure of each separate product under the figure used in obtaining it.*

Examples. 1.

SOLUTION. Having written the multiplier under the unit of the multiplicand, multiply the 5 units by 7, obtaining 35. Then set down the 5 units directly under the 7 and carry the 3; in other words, reserve the 3 tens for the tens column. Next multiply the seven tens by 7, obtaining 49, and add the 3 which is carried, and obtain 52 tens (which is the same as 5 hundreds and 2 tens). Set down 2 tens and carry the 5 hundreds; multiply 1 and 7 and add the 5 which was carried, making 12, which can be written down in full. The product then reads, 1,225.

$$\begin{array}{r} 175 \text{ multiplicand} \\ 7 \text{ multiplier} \\ \hline 1225 \text{ product} \end{array}$$

2. Find the product of 145 and 13.

SOLUTION a. It can be seen that $13 = 10 + 3$, hence the product of 13×145 will be the same as $(10 \times 145) + (3 \times 145)$. Adding the products gives 1,885 as a result.

$$\begin{array}{r} 145 \times 3 = 435 \\ 145 \times 10 = 1450 \\ \hline 1885 \end{array}$$

SOLUTION b. The same result is obtained, however, if the numbers are arranged as follows:

Commence with 3, multiply through and write the product 435. Under this write the product 1,450 obtained by multiplying by 10. In this latter product the 0 may be discarded but it must be remembered to write the 5 under second place. Adding these two products, called *partial products*, gives the final product 1,885.

$$\begin{array}{r} 145 \\ 13 \\ \hline 435 \\ 145 \\ \hline 1885 \end{array}$$

3. Multiply 1246 by 235.

SOLUTION. Note that the three partial products are the results of multiplying by 5, 3, and 2 where each successive partial product is set one place further to the left than the preceding one. Note that 0 is under 5, 8 is under 3, and 2 is under 2; in other words, the first figure of each partial product is placed under the digit used to obtain it.

$$\begin{array}{r} 1246 \\ 235 \\ \hline 6230 \\ 3738 \\ 2492 \\ \hline 292810 \end{array}$$

Two special cases not covered by the general rules given above should be here considered.

Case 1. When digits of multiplier are separated by ciphers:

Example. Find the product of 13,456 and 12,004.

SOLUTION. Although the multiplier contains four digits, in the short method only two partial products appear.

The first figures obtained by multiplying by 4 and 2 appear under these respective digits, the zeros simply marking the absence of any other characters in the product.

Regular method

$$\begin{array}{r} 13456 \\ 2004 \\ \hline 53824 \\ 00000 \\ 00000 \\ 26912 \\ \hline 26965824 \end{array}$$

Short method

$$\begin{array}{r} 13456 \\ 2004 \\ \hline 53824 \\ 26912 \\ \hline 26965824 \end{array}$$

Case 2. When ciphers are at right of multiplier or multiplicand:

Example. Multiply 5,760 by 3,000.

SOLUTION. In this case it is necessary only to multiply 576 by 3 giving 1,728; then annex the total number of ciphers found at the right of both multiplier and multiplicand, in this case four, giving as the final result, 17,280,000.

$$\begin{array}{r} 5760 \\ 3000 \\ \hline 17280000 \end{array}$$

16. Short Methods. *To multiply by ten and *powers of ten.* This method is only a slight variation of that given in the previous paragraph.

*A power of ten is the product obtained by using ten as a factor a certain number of times. Thus, 1,000 and 10,000 are, respectively, the third and fourth powers of 10.

Annexing a cipher to a whole number multiplies that number by ten, etc. Thus, to multiply 378 by 1,000 write at once 378,000.

To multiply by a number a little less than 10, 100, or 1000 the process may be shortened as shown in the following:

Examples. 1. Multiply 254 by 99.

SOLUTION. 99 is 100 diminished by 1; hence, multiply 254 first by 100 and then by 1 and subtract the results.

$$\begin{array}{r} 25400 \\ 254 \\ \hline 25146 \end{array}$$

2. Multiply 196 by 997.

SOLUTION. 997 is 1,000 diminished by 3, hence, multiply 196 first by 1,000 and then by 3 and subtract the results.

$$\begin{array}{r} 196000 \\ 588 \\ \hline 195412 \end{array}$$

To multiply by 25 annex two ciphers to the multiplicand and divide by 4.

To multiply by 11:

(a) When the multiplicand contains two figures, place their sum between them. If this sum is greater than 10, carry 1 to the third place.

Example. Multiply 47 by 11.

SOLUTION. Place 7 in units place. Add 4 and 7. Putting 1 in tens place and carrying 1. Place (1 + 1) in hundreds place. Result 517.

(b) When the multiplicand is any number, write the right-hand figure in units place; then add the first and second, second and third, and so on, finally setting down the left-hand figure. Carry as usual.

Example. Multiply 365 by 11.

SOLUTION. Write 5; $5 + 6 = 11$; write 1 and carry 1; 1 carried + 6 + 3 = 10; write 0; 1 carried + 3 = 4; write 4, making the result, 4,015.

$$\begin{array}{r} 365 \\ 11 \\ \hline 4015 \end{array}$$

17. A Method of Checking Multiplication. Add separately the figures of the multiplicand and multiplier until they are reduced to one figure each; then multiply these together and again reduce this product by addition to one figure. If the multiplication is correct, the final result will usually check with the successive additions of the figures of the product.

Example. Multiply 6,547 by 301 and check.

SOLUTION. The solution is shown in the margin. To check proceed as follows: Taking the multiplicand, $6 + 5 + 4 + 7 = 22$; $2 + 2 = 4$. Again for the multiplier, $3 + 1 = 4$. Their product, $4 \times 4 = 16$; $1 + 6 = 7$. Treating the product in the same manner; $1 + 9 + 7 + 6 + 4 + 7 = 34$; $3 + 4 = 7$. The multiplication is correct.

6547	— 4
301	— 4
6547	16—7
19641	
1970647	— 7

18. Letters may be used in multiplication as follows: The product of a and b is $a \times b$, and the product of a , b , and c is $a \times b \times c$. If $a = 6$, $b = 4$, and $c = 1$, the product of a , b , and c is $6 \times 4 \times 1$ or 24. The common practice is to omit the multiplication sign when letters are multiplied by letters or numbers thus: $2 \times x$ is written $2x$; $2a \times 3b$ is written $6ab$; $3a \times 4b \times 5c$ is written $60abc$. The order in which the letters are placed is immaterial although the alphabetical order is usually followed. For example, $60 cab$ is the same as $60abc$.

PROBLEMS FOR PRACTICE

Multiply:

- | | |
|--|--------------------|
| 1. 2,928 by 364. | Ans. 1,065,792 |
| 2. 7,319 by 394. | |
| 3. 5,698 by 792. | Ans. 4,512,816 |
| 4. $9a$ by $2b$. | Ans. $18ab$ |
| 5. $3ab$ by $4cd$. | Ans. $12abcd$ |
| 6. 3,186 by 839. | Ans. 2,673,054 |
| 7. 42,308 by 692. | |
| 8. 876 by itself. | Ans. 767,376 |
| 9. $4ab$ by $14c$. | |
| 10. 57 by 1,000. | |
| 11. 52 by 99. | |
| 12. 16 by 25. | |
| 13. 92 by 11. | |
| 14. 103 by 25. | |
| 15. There are 746 watts in a horse-power. How many watts are there in 20 horse-power? (Suggestion: Multiply 746 by 20.) | |
| 16. A piston has an area of approximately 113 square inches. If the steam pressure is 47 pounds per square inch, what is the total pressure upon it? | |
| | Ans. 5,311 pounds. |

17. The head of a boiler has an area of 11,310 square inches. If the pressure per square inch is 40 pounds, what is the total pressure on the head? Ans. 452,400 pounds.

18. A concrete mixer delivers 14 cubic yards per hour. A gang of men takes away 12 cubic yards per hour. How many cubic yards will remain unmoved at the end of 4 hours? At the end of 8 hours?

19. If there are 12 threads per inch on a screw, how many threads are there in 4 inches?

20. If a piston moves through 468 feet in one minute, how far does it travel in 45 minutes? Ans. 21,060 feet.

21. If a boiler evaporates 1,945 pounds of water in one hour, how many pounds will it evaporate in 9 hours?

22. A certain girder supports 136,925 pounds. How much will 65 such girders support? Ans. 8,900,125 pounds.

23. An engine in a certain power plant requires 18 pounds of steam per horse-power per hour. If the engine is developing 640 horse-power, what is the total steam consumption?

DIVISION

19. *Division* is a process of finding how many times one quantity contains another. In division there are three principal terms, the dividend, the divisor, and the quotient or answer.

The *dividend* is the quantity to be divided.

The *divisor* is the quantity which is divided into the dividend.

The *quotient* is the number of times the divisor is contained in the dividend.

When the dividend does not contain the divisor an exact number of times, the excess is called the *remainder*. The remainder being a part of the dividend will always be of the same kind as the dividend and must necessarily be less than the divisor.

Division may be indicated in any of the following ways: $24 \div 2$; $\frac{24}{2}$; $2\overline{)24}$

Division is the reverse of multiplication, as shown by the following:

$$6 \times 7 = 42$$

$$42 \div 6 = 7$$

$$42 \div 7 = 6$$

$$5 \times 8 = 40$$

$$40 \div 8 = 5$$

$$40 \div 5 = 8$$

20. There are two distinct methods used, viz, *long division*

and short division; in the former all the work is written out but in the latter the process is performed mentally and the result only is written. Short division is generally used when the divisor does not exceed 12

The following examples illustrate the two processes.

Examples. 1. Divide 720 by 5.

SOLUTION. In long division it is found that 5 is contained in 7 once. Write 1 as the first figure of the quotient, and subtract, giving a remainder of 2. To the remainder *annex the next figure of the dividend, and divide as before, obtaining 4 as the second figure of the quotient. Annex 0 which is the next figure of the dividend, and divide again by 5, obtaining 4 as the last figure of the quotient with no remainder. The division is now complete.

$$\begin{array}{r} \text{Long Division} \\ 144 \text{ quotient} \\ 5 \overline{)720} \\ \underline{5} \\ 22 \\ \underline{20} \\ 20 \\ \underline{20} \end{array}$$

$$\begin{array}{r} \text{Short Division} \\ 5 \overline{)720} \\ 144 \text{ quotient} \end{array}$$

It often happens, after bringing down a figure from the dividend, that the number is too small to contain the divisor. In this case place a zero in the quotient, and continue bringing down the figures from the dividend until the number thus formed will contain the divisor. The following example illustrates this point:

2. Divide 10,426 by 13.

SOLUTION. It is seen that the first two places in the dividend are less than the divisor; therefore, three places must be taken. 13 is contained in 104 exactly 8 times, giving 8 as the first figure in the quotient. There being no remainder, the next figure of the dividend, when brought down, stands alone and is, of course, less than 13. Place a zero as the second figure of the quotient and annex the next figure 6 of the dividend making 26, into which 13 goes exactly twice. This makes the complete quotient 802 without a remainder.

$$\begin{array}{r} 802 \\ 13 \overline{)10426} \\ \underline{104} \\ 26 \\ \underline{26} \end{array}$$

21. Rules for Long Division. (a) *Write the divisor and dividend in the order named, and draw a curved line between them.*

(b) *Find how many times the divisor is contained in the left-hand*

*In the above solution the term "annex", meaning to place after, is used. The opposite term "prefix", meaning, to place before, is also often used. Be careful to make a distinction between "annex" and "add".

figure or figures of the dividend, and write the number in the quotient over the dividend.

(c) *Multiply the divisor by this figure of the quotient, writing the product under that part of the dividend from which it was obtained; subtract, and to the remainder annex the next figure of the dividend.*

(d) *Find how many times the divisor is contained in the number thus formed, and write the figure denoting it at the right hand of the last figure of the quotient.*

(e) *Proceed in this manner until all the figures of the dividend are divided. If there is a remainder after dividing all the figures of the dividend, place the remainder over the divisor with a line between them, and annex to the quotient.*

(f) *The proper remainder is, in all cases, less than the divisor. If, in the course of the operation, it is found to be larger than the divisor, this indicates that there is an error in the work and that the figure in the quotient should be increased.*

Example. Divide 5,441 by 26.

SOLUTION:

$$\begin{array}{r} 209\frac{7}{8} \\ 26 \overline{) 5441} \\ \underline{52} \\ 241 \\ \underline{234} \\ 7 \text{ remainder} \end{array}$$

Proof. In order to prove division, multiply the quotient by the divisor, and add the remainder, if there is any. If the quantity thus obtained gives the dividend, the work is correct.

22. *Short Methods. *To divide by ten and powers of ten.* From the right in the dividend point off as many places as the divisor contains ciphers. The figures so cut off represent the remainder, to be written over the divisor and annexed to the quotient.

To divide by multiples of ten.

Example. Find the quotient of $18,653 \div 3,000$.

SOLUTION. Mark off as many figures in the dividend as there are ciphers in the divisor, thus dividing by 1,000. This leaves 18 to the left of the mark to be divided by 3, giving a quotient of 6 with a remainder of 653. Place this remainder over the divisor and annex it to the 6, giving, finally, $6\frac{653}{3000}$.

$$\begin{array}{r} 6 \frac{653}{3000} \\ 3 \overline{) 00018653} \\ \underline{18} \\ 653 \end{array}$$

*These methods may be made more comprehensive and valuable after the subject of Decimals has been discussed.

To divide by 25.

Multiply the dividend by 4 and divide the product by 100, by cutting off two figures to the right.

23. Letters may be used in division as follows: a divided by b may be expressed as $a \div b$ or $\frac{a}{b}$; a divided by c , as $a \div c$, or $\frac{a}{c}$. If $a = 6$ and $b = 3$, a divided by b will equal $6 \div 3 = 2$. If $a = 6$ and $c = 2$, a divided by c will equal $6 \div 2$ or 3. In dividing terms containing letters, it is useful to remember the statement on page 20, that *division is the reverse of multiplication*. For example, $4abc$ divided by $2c$ equals $2ab$, for $2c \times 2ab$ equals $4abc$.

Examples. 1. Divide $16abcd$ by $4ad$.

SOLUTION. The coefficient of the divisor is contained in the coefficient of the dividend 4 times. Place a 4 in the quotient. Again a and d of the divisor are contained in a and d of the dividend exactly once but ad is contained in $abcd$, bc times for $ad \times bc$ equals $abcd$. The complete quotient is, therefore, $4bc$ without a remainder.

$$\begin{array}{r} 4bc \\ 4ad \overline{) 16abcd} \\ \underline{16abcd} \end{array}$$

2. Divide $(15abc + 6ab)$ by $3b$.

SOLUTION. In this problem the two terms of the dividend are divided separately and the two quotients connected by the proper sign. $15abc$ divided by $3b$ gives a quotient of $5ac$; $6ab$ divided by $3b$ gives $2a$ as a quotient. The final quotient is, therefore, $5ac + 2a$.

$$\begin{array}{r} 5ac + 2a \\ 3b \overline{) 15abc + 6ab} \\ \underline{15abc + 6ab} \end{array}$$

PROBLEMS FOR PRACTICE

Divide:

1. 65,814 by 6.

Ans. 10,969

2. $6ab$ by $3b$.

Ans. $2a$

3. 3,870 by 10.

4. $24abcd$ by $6d$.

Ans. $4abc$

5. 9,473 by 100.

Ans. $94\frac{73}{100}$

6. 13,987 by 1000.

7. $30abcd$ by $5ac$.

8. $(18abc + 12abd)$ by $3b$.

9. If coal costs 5 dollars per ton, how many tons may be bought for 275 dollars? (Suggestion: Divide 275 by 5.)

10. A steamer runs 276 miles in 23 hours. What is her average speed per hour? (Suggestion: Divide 276 by 23.)

11. How many 30-foot rails are there in a double track (four rails) two miles long? There are 5,280 feet in one mile. (Suggestion: Divide $2 \times 4 \times 5,280$ by 30.)

12. If 9 days' work will pay for 6 tons of coal at 6 dollars per ton, what is the price of a day's labor?

13. A company furnishes equal power to 26 establishments. The total horse-power is 8,450. How much does each receive?

14. If a locomotive goes to shop for inspection after every average run of 283 miles, how many times would it be in shop during runs aggregating 10,188 miles? Ans. 36 times.

15. A certain boiler supplies steam for heating. If there are 180 square feet of heating surface in the boiler and each radiator requires 12 square feet of heating surface of the boiler, how many radiators can be supplied by the boiler? Ans. 15 radiators.

16. If 800 cubic feet of air are required for each person, how many people can occupy a room that contains 21,600 cubic feet?

17. If 1 foot of 1 inch pipe is allowed for every 90 cubic feet of space in heating a factory, how many feet of the same pipe will be required to heat 225,000 cubic feet of space?

FACTORING

24. An *integer* is a whole number, as 1, 8, 15.

All numbers are either odd or even. An *odd* number is a number that cannot be divided by 2 without a remainder, as 3, 9, 13, etc. An *even* number is one that can be divided by 2 without a remainder, as 4, 6, 8, etc.

A *prime* number is a number which can be exactly divided only by itself or 1, as 1, 3, 5, 7, etc.

A *factor* of an integer is a number which is contained an exact number of times in the given integer.

A *prime factor* is a factor which is a prime number.

Numbers are prime to each other when they have no factor in common. Thus 7 and 11 are prime to each other; also 18 and 25, etc.

In speaking of the factors of a number it is not customary to include the number itself or 1. For this reason a prime number is said to have no factors.

Example. What are the prime factors of 16?

SOLUTION. Divide 16 by 2 which is the least prime number greater than 1, and obtain 8 as the quotient. Since 8 may again be divided by 2, the division is carried out giving 4 as the next quotient. Next divide 4 by 2 giving a quotient of 2, which is a prime number and cannot be further divided. Now the several divisors and the last quotient, all of which are prime, are the prime factors of 16.

$$\begin{array}{r} 2 \overline{)16} \\ \underline{2} \\ 8 \\ 2 \overline{)8} \\ \underline{2} \\ 4 \\ 2 \overline{)4} \\ \underline{2} \\ 2 \end{array}$$

The rule for finding prime factors may be expressed as follows:

Divide the given number by the least prime number (greater than 1) that will divide it; and divide each quotient, in the same manner. Continue dividing until a prime number is obtained as the quotient. The several divisors and the last quotient will be the prime factors desired.

Example. What are the prime factors of 78?

SOLUTION.

$$\begin{array}{r} 2 \overline{)78} \\ \underline{2} \\ 39 \\ 3 \overline{)39} \\ \underline{3} \\ 13 \end{array} \quad \text{Ans. 2, 3, and 13}$$

25. Factoring by Observation. Certain numbers have such characteristics that some of their factors may be found without the trouble of an actual operation in division. The simplest of these rules for factoring by observation are as follows:

2 is a factor of all even numbers.

3 is a factor of all numbers the sum of whose digits is divisible by 3.

4 is a factor of all numbers whose last two places are divisible by 4.

5 is a factor of all numbers ending in 5 or 0.

10 is a factor of all numbers whose last figure is 0.

Examples. 1. Factor the number 56.

SOLUTION. By inspection it is seen that 56 is divisible by 4, and although 4 is not a prime factor, its use saves one division. The resulting factors are 4, 2, and 7

$$\begin{array}{r} 1 \overline{)56} \\ \underline{1} \\ 11 \\ 1 \overline{)11} \\ \underline{1} \\ 1 \end{array} \quad \text{7 Ans. 4, 2, and 7}$$

2. Factor the number 720.

SOLUTION. According to the rule in §25 this number is divisible by 10 and, although 10 is not a prime number, the process of factoring is shortened by its use. The quotient 72 is divisible by 4, which should be used instead of dividing twice by the factor 2. The remaining factors are 2, 3, and 3. 720 may also be divided into the factors 10, 8, and 9, the two latter being equal, respectively, to 4×2 and 3×3 .

$$\begin{array}{r} 10 \overline{)720} \\ \underline{10} \\ 72 \\ 4 \overline{)72} \\ \underline{4} \\ 18 \\ 2 \overline{)18} \\ \underline{2} \\ 9 \\ 3 \overline{)9} \\ \underline{3} \\ 3 \end{array} \quad \begin{array}{l} \text{3 Ans. 10, 4, 2, 3,} \\ \text{and 3} \end{array}$$

3. Factor the number 426.

SOLUTION.

$$2 \overline{)426}$$

$$3 \overline{)213}$$

71 Ans. 2, 3, and 71

Find the prime factors of the following:

1. 36

Ans. 2, 2, 3, and 3

2. 30

3. 68

4. 148

5. 189

Ans. 3, 3, 3, and 7

6. 1,264

7. 2,552

8. 1,932

9. 91

10. 2,508

Ans. 2, 2, 3, 11, 19

CANCELLATION

26. Cancellation Methods. When a series of multiplied factors is to be divided by a second series, the operation may be shortened by a process called "Cancellation". The first series is placed above and the second below a horizontal line and divisions are performed between the factors on opposite sides of the line. Any number on one side which is exactly divisible by a number on the other side of the line may be so divided.

Examples. 1. Divide 18×5 by 10×3 .

SOLUTION: Arranging the first series above and the second series below the line we have

$$\begin{array}{ccccccc} & & 6 & & 6 & 1 & \\ 18 \times 5 & = & \cancel{18} \times 5 & = & \cancel{18} \times \cancel{3} & & \\ 10 \times 3 & = & 10 \times \cancel{3} & = & \cancel{10} \times \cancel{3} & & \\ & & 1 & & 2 & 1 & \end{array}$$

It will be seen that 18, the first term above, and 3, the second term below the line, are divisible by 3. The 18 is, therefore, crossed out or "cancelled out" and a 6 placed above it; the 3 below is also cancelled and a 1 placed below it. Similarly 5 is contained in 5 and 10, the remaining factors, and these should also be cancelled out as indicated, a 1 being placed above the 5 and a 2 below the 10.

Further inspection shows that the 6 which remains above the line is divisible by the 2 which remains below the line and this can-

cancellation gives a final result of 3. The complete process is

$$\begin{array}{r} 3 \\ 6 \quad 1 \\ 18 \times 5 = 3 \\ 10 \times 3 = 1 \\ 2 \quad 1 \\ 1 \end{array} = 3$$

2.
$$\frac{6 \times 8 \times 12 \times 18 \times 24}{2 \times 3 \times 2 \times 4 \times 6} = ?$$

SOLUTION. To cancel, select any two numbers one above and one below the line, and find some number that will divide each of them without a remainder. For instance, it is seen that 2 will be contained an exact number of times in both 6 and 2. Then perform the division, crossing out both numbers and placing the results directly over and below, the numbers crossed out. Proceed in this manner until there is no longer a number below the line that can be cancelled with one above the line. Then multiply together all the numbers above the line and use as a dividend; and multiply together those below the line, and use as a divisor.

$$\begin{array}{cccccc} 3 & 4 & 4 & 3 & 6 \\ 6 \times 8 \times 12 \times 18 \times 24 = \\ 2 \times 3 \times 2 \times 4 \times 6 \\ 1 & 1 & 1 & 1 & 1 \end{array}$$

$$\frac{3 \times 4 \times 4 \times 3 \times 6}{1} = 864$$

3.
$$\frac{16 \times 2 \times 15 \times 4}{32 \times 6 \times 8 \times 22} = ?$$

SOLUTION. In an example where the product above the line, after cancelling, is less than the product below the line, the result is allowed to stand as obtained, thus. Since 5 and 22 do not cancel into any of the other numbers the result is the product of the quantities in the numerator divided by the product of the quantities in the denominator.

$$\begin{array}{cccc} 1 & 1 & 5 & 1 \\ 16 \times 2 \times 15 \times 4 = \frac{5}{88} \\ 32 \times 6 \times 8 \times 22 \\ 2 & 3 & 2 & \\ & 1 & & \end{array}$$

4. Four men have the task of sorting 24 boxes of castings, each box containing 720 pieces. If each man can handle 12 pieces per minute, how many hours will it take the men to sort the castings?

SOLUTION. A little examination of this problem will show that the total number of pieces to be handled will form the quantity above the line, that is, 24

times 720. As there are 4 men and each can handle 12 pieces per minute, the number handled per minute will be 4×12 and the number per hour will be $4 \times 12 \times 60$. This quantity is placed below the line as the total number of pieces divided by the number handled per hour will give the required result. Carrying on the cancellation as before, the final result is 6, that is, it will take the men 6 hours to sort the castings.

$$\begin{array}{r} 1 \\ 12 \\ 6 \quad 72 \\ \hline 24 \times 720 \\ 4 \times 12 \times 60 \\ \hline 1 \quad 1 \quad 6 \\ 1 \end{array} = 6$$

Rules. *Cancel the common factors from both dividend and divisor.*

Next, divide the product of the remaining factors of the dividend by the product of the remaining factors of the divisor.

It is seen, therefore, that cancellation is merely a combination of the processes of factoring and division.

PROBLEMS FOR PRACTICE

Divide:

1. $2 \times 3 \times 8 \times 12 \times 24$ by $6 \times 4 \times 36 \times 4$. Ans. 4
2. $18 \times 24 \times 32 \times 36$ by $9 \times 18 \times 4 \times 18$.
3. $15 \times 20 \times 25 \times 27$ by $15 \times 18 \times 25 \times 10$. Ans. 3
4. $40 \times 48 \times 54 \times 60$ by $30 \times 24 \times 72 \times 3$.
5. $12 \times 60 \times 36 \times 70$ by $28 \times 5 \times 48 \times 6$. Ans. 45
6. $32 \times 36 \times 33 \times 45$ by $24 \times 30 \times 44 \times 9$.
7. 18 piers, each consisting of 5 piles, were set by 10 men in 3 days. What is the cost of driving each pile if each of the men receive 3 dollars a day?
8. Each of 40 teamsters hauls 9 yards of sand per day for 4 days at 1 dollar per yard. How many men with wheelbarrows will earn the same amount in two days wheeling 3 yards per day at 1 dollar per yard? Ans. 240 men.

REVIEW PROBLEMS

1. A steam plant has two engines of 934 horse-power each. The starboard engine of a steamship develops 3,218 horse-power and the port engine 3,232 horse-power. How much greater power is developed in the ship than in the steam plant? **Ans. 4,582 H. P.**

2. A pump delivers 33,720 cubic feet of water in two hours (60 minutes each). How many cubic feet are delivered per minute? **Ans. 281 cu. ft.**

3. A nine-foot blower makes 175 revolutions per minute. How many revolutions will it make in 29 minutes? **Ans. 5,075 rev**

4. There are 360 rivets in a hundred pounds. How many

pounds will be required to afford 64,800 rivets. Ans. 18,000 pounds.

5. The heating surface in a locomotive is 88 square feet in the fire box and 792 square feet in the tubes. What is the total heating surface? How many times as much surface is there in the tubes than in the fire box?

Ans. $\left\{ \begin{array}{l} 880 \text{ square feet} \\ 9 \text{ times} \end{array} \right.$

6. The weight of a battery of eight marine boilers and their equipments is as follows:

Boilers complete with mountings.....	295 tons
Water in boilers.....	73 "
Funnels.....	50 "
Stoke-hole plates, floors, etc.....	23 "
Feed pumps.....	7 "
Fans and fan engines.....	8 "
Feed regulators.....	2 "
Tools and fittings.....	2 "
Spare gear.....	10 "

What is the weight of two such batteries?

7. At 100 degrees Fahrenheit a cubic foot of water weighs 62 pounds. At 205 degrees a cubic foot weighs 60 pounds. What is the difference in weight between 173 cubic feet of water at 100 degrees and 189 cubic feet at 205 degrees? Ans. 614 pounds.

8. A roof is composed of 11 frames. The weight of one frame in detail is as follows:

2 rafters each weighing.....	875 lbs.
5 rods, each weighing.....	176 "
16 bolts, each weighing.....	5 "
8 bridle-straps, each weighing.....	15 "
2 piers supporting rafters at ridge, average each. . .	11 "
6 pieces at foot of struts, average each.....	11 lbs.
4 pieces uniting rafters at junction in strut, with bolts and nuts, each.....	44 "
2 rafter shoes, each.....	144 "
2 cast-iron struts, each.....	154 "

What is the total weight of the roof? Ans. 40,590 pounds.

9. A double mine ventilating fan runs at the rate of 84 revolutions per minute. It gives 2,818 cubic feet of air per revolution.

How many cubic feet should it give per minute?

Ans. 236,712 cubic feet.

10. If oak-tanned leather belting costs 2 dollars per foot, and four-ply stitched canvas belting costs 1 dollar per foot, what is the difference in the cost of 40 feet? Ans. 40 dollars.

11. If there are 9,326 heat units in a pound of lignite coal, how many heat units in 287 pounds?

12. If 23 square feet of No. 30 sheet iron weighs 11 pounds, how much will 23 square feet weigh if three times as thick?

13. The weight of the masonry of a bridge and an engine passing over it is 1,698,575 pounds. The engine weighs 198,560 pounds. What is the weight of the masonry?

14. A pound of Pennsylvania petroleum will theoretically evaporate about 22 pounds of water. How many pounds are necessary to evaporate 4,378 pounds of water?

15. A cubic foot of hemlock weighs 25 pounds. A cubic foot of iron weighs 450 pounds. Find the difference between the weight of 230 cubic feet of hemlock and 87 cubic feet of iron.

16. Three guy ropes are fastened to a stake. The pull on one rope is 560,800 pounds; the pull on the second is 118,421 pounds; and on the third is 104,863 pounds. What is the total pull?

17.* Divide $(20ac \times 4bd)$ by $16b$. Ans. $5acd$

18.* Divide the product of $7a$ and $6bcd$ by $3ad$.

19. Let a , b , and c have, respectively, the values 4, 7, and 2. Subtract a from the sum of b and c and give the numerical result.

Ans. 5

20. Give values of 5, 9, 6, and 12 to a , b , c , and d , respectively; find the numerical result of adding $3a$, $7c$, and $5d$, and subtracting $8b$ from this sum.

21. The record of a pieceworker in a shop for one week was as follows: for Monday 122 pieces, Tuesday 140, Wednesday 114, Thursday 151, Friday 132, and Saturday 142. What is his average day's work. (Suggestion: An average is found by dividing the sum of the various items by the number of items.)

Ans. 134 pieces per day.

* Work this problem by cancellation.

PRACTICAL MATHEMATICS

PART II

FRACTIONS

27. A *fraction* is an expression denoting one or more equal parts of a unit, and may be regarded as indicating division. The fraction is written, *i. e.*,* the division is indicated, by placing the dividend over the divisor with a line between.

Thus, $\frac{5}{12}$ denotes that 5 of the 12 equal parts of a unit are to be taken. $\frac{5}{12}$ is merely a different way of writing $5 \div 12$, and the fraction is used because the dividend is less than the divisor.

To say $\frac{1}{4}$ of a 100 pound keg of bolts indicates, that the keg of bolts is to be divided into 4 equal parts, and 1 of these parts taken. In this use, the unit may be considered as made up of a number of equal parts, but when used separately and without reference to any certain thing, it is convenient to consider the fraction merely as an unperformed division; thus, $\frac{1}{4} = 1 \div 4$; $\frac{3}{4} = 3 \div 4$; $\frac{875}{361} = 875 \div 361$.

The quantity below the line is called the *denominator*. It shows into how many equal parts the unit is divided.

The quantity above the line is called the *numerator*. It shows how many of these equal parts are taken. Thus, in the fraction $\frac{1}{12}$, the 12 shows that the unit has been divided into 12 equal parts, and the 1 shows that one of the 12 parts is taken. In the fraction $\frac{a}{b}$ the unit is divided into b parts and a of these parts are taken.

The numerator and denominator are called the *terms* of a fraction.

* The letters "i. e." are used to represent the expression "that is".

A *fractional unit* is one of the equal parts into which the unit in question is divided. Thus, if the unit is divided into fourths, $\frac{1}{4}$ is the fractional unit; if divided into b parts, $\frac{1}{b}$ is the fractional unit.

Since any number divided by 1 gives a quotient equal to the dividend, any whole number may be expressed as a fraction by writing 1 for the denominator. Thus, 4 may be written $\frac{4}{1}$, as $\frac{4}{1} = 4 \div 1 = 4$.

The *value* of a fraction depends upon the value of the fractional units and the number of these units taken, or simply upon the division of the numerator by the denominator. Thus, in $\frac{4}{2}$, the quotient of $4 \div 2$ is 2, and the value of the fraction can be expressed as 2.

If the numerator and the denominator are equal, the value of the fraction is 1. Thus, $\frac{8}{8}$ may be expressed as $8 \div 8$ and is equal to 1. This shows that one unit is divided into eight parts, each part being an eighth and that 8 of these are taken, making a unit or 1.

Strictly speaking a fraction is less than a unit; hence if the numerator is less than the denominator, the value is less than 1, and it is known as a *proper fraction*. For example, the expression $\frac{8}{9}$ means that a unit has been divided into nine parts, each part forming a fractional unit having a value of $\frac{1}{9}$, and that eight of these parts are taken. This is one less than the nine parts necessary to make a unit, and therefore $\frac{8}{9}$ is less than 1.

If the numerator is greater than the denominator, the value of the fraction is greater than 1, and it is called an *improper fraction*. Thus, the fraction $\frac{8}{7}$ is an improper fraction, because 7 is contained once in 8 with a remainder, or, expressing it in another way, because eight parts, each one being a seventh of a unit, have been taken, forming a unit and one seventh.

REDUCTION OF FRACTIONS

28. *To reduce a fraction* is to change its form without changing its value.

To reduce a fraction to higher terms multiply both numerator and denominator by the same quantity. Thus, $\frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12}$. The value of the fraction has been increased three times by multiplying the numerator by 3, and then decreased just as many times by multiplying the denominator by 3. If in the fraction $\frac{b}{d}$ both the numerator and denominator are multiplied by c , the value of the fraction is not changed but the form is changed to $\frac{bc}{dc}$. Thus, multiplying both numerator and denominator by the same quantity does not change the value of the fraction.

To reduce a fraction to lower terms divide both numerator and denominator by the same number. Thus, $\frac{4}{6} \div \frac{2}{2} = \frac{2}{3}$. In this case dividing the denominator by 2 changes the fractional parts from sixths to thirds, which are twice as large as sixths and this much of the operation has doubled the value of the fraction. Dividing the numerator by 2 decreases the number of parts to one-half of the original number. Therefore dividing both numerator and denominator by the same quantity does not change the value of the fraction. In the same way $\frac{mp}{mn}$ may be reduced to $\frac{p}{n}$ by dividing both numerator and denominator by m .

A fraction is reduced to its lowest terms when its numerator and denominator have no common factor other than 1; that is when the terms are prime to each other. Thus, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{15}{16}$, $\frac{a}{b}$, are reduced to their lowest terms, but $\frac{4}{6}$ is not, as 4 and 6 may both be divided by 2, reducing the fraction to $\frac{2}{3}$, and $\frac{ad}{ac}$ is not, as ad and ac may both be divided by a , reducing the fraction to $\frac{d}{c}$.

An *entire quantity* is one which has no fractional part as 3, 11, or *b*. An entire quantity, which is also a number, is called an *integer*. For example, 3 and 11 are integers.

A *mixed quantity* consists of an entire quantity and a fraction. Thus, $5\frac{1}{6}$, $3\frac{7}{8}$, and $a + \frac{b}{d}$ are mixed quantities. To reduce a mixed quantity to an improper fraction, multiply the entire quantity by the denominator of the fraction, add to this product the numerator of the fraction and place their sum over the denominator. To reduce $3\frac{7}{8}$ to an improper fraction, reduce 3 units to eighths by multiplying 3 by 8 obtaining 24; add this to 7 and the result, 31, is the numerator of the fraction. $3\frac{7}{8} = \frac{31}{8}$. In the same way $a + \frac{b}{d} = \frac{ad + b}{d}$

To reduce an improper fraction to a mixed quantity divide the numerator by the denominator, write the quotient as the entire quantity of the mixed quantity and place the remainder over the divisor (denominator of improper fraction) as the fraction.

Examples. 1. Change $\frac{31}{21}$ to a mixed number.

SOLUTION. Divide the numerator 31 by the denominator 21, and place the remainder 10 over the divisor 21 as the fraction of the mixed quantity. As 21 twenty-firsts constitute one unit, 31 twenty-firsts constitute one unit and ten more twenty-firsts.

$$\begin{array}{r} 1\frac{10}{21} \text{ quotient} \\ 21 \overline{)31} \\ \underline{21} \\ 10 \text{ remainder} \end{array}$$

2. Reduce $\frac{35ab}{16a}$ to a mixed number.

SOLUTION. Following the process already given $\frac{35ab}{16a} = 2b + \frac{3b}{16}$, for by cancellation $\frac{3ab}{16a}$ becomes $\frac{3b}{16}$, giving the final form $2b + \frac{3b}{16}$. This quantity $2b + \frac{3b}{16}$ can be changed to the original quantity thus: Following the rule given above, $16 \times 2b$ equals $32b$ and adding to this $3b$ and placing 16 as the denominator we have $\frac{35b}{16}$. Multiplying both numerator and denominator by the same quantity *a* does not change its value and gives $\frac{35ab}{16a}$, the original fraction.

$$\begin{array}{r} 2b + \frac{3b}{16} \text{ quotient} \\ 16a \overline{)35ab} \\ \underline{32ab} \\ 3ab \text{ remainder} \end{array}$$

* This must not be expressed as $2b \frac{3b}{16}$, as this would mean $2b \times \frac{3b}{16}$. Page 15. Part I.

PROBLEMS FOR PRACTICE

Reduce to mixed quantities:

1. $\frac{25679}{4}$

2. $\frac{13}{7}$

3. $\frac{117}{5}$ Ans. $23\frac{2}{5}$

Change to improper fractions:

4. $3\frac{3}{5}$

5. $90\frac{3}{9}$

6. $21\frac{7}{20}$

Reduce to lowest terms:

7. $\frac{369}{936}$

8. $\frac{288}{360}$

Find the proper numerators:

9. $\frac{1}{16} = \frac{?}{96}$

10. $\frac{3}{5} = \frac{?}{525}$

11. $\frac{21}{22} = \frac{?}{462}$ Ans. $\frac{441}{462}$

SUGGESTION: Problem 9 $96 \div 16 = 6$ Hence, 6 is the multiplying factor
for numerator and denominator of fraction See § 28, second paragraph.

Reduce to mixed quantities:

12. $\frac{14abc}{5bc}$ Ans. $2a + \frac{4a}{5}$

13. $\frac{16a}{6}$

Reduce to lowest terms:

14. $\frac{8a}{12}$ Ans. $\frac{2a}{3}$

15. $\frac{16bc}{32c}$

Change to improper fractions:

16. $4 + \frac{6}{7p}$ Ans. $\frac{28p+6}{7p}$

17. $7a + \frac{3a}{5}$

Find the proper numerators:

18. $\frac{1}{5a} = \frac{?}{25a}$

19. $\frac{5}{6b} = \frac{?}{12b}$ Ans. $\frac{10}{12b}$

LEAST COMMON DENOMINATOR

29. The product of a number of factors is called a *multiple* of any one of them. Thus, 6 is a multiple of 3; 21 is a multiple of 7; and ab is a multiple of b .

Now the denominators of any group of fractions must be reduced to some common multiple, called a *common denominator*, before the fractions may be added, subtracted, or compared in value. This

common denominator may be obtained by multiplying together the denominators of the fractions. Suppose, for example, it is desired to reduce $\frac{1}{3}$ and $\frac{1}{4}$ of a foot to fractional parts of a foot, having a common denominator. Let the common denominator be $3 \times 4 = 12$. To reduce $\frac{1}{3}$ to higher terms, *i. e.*, to a fraction having the given denominator 12, multiply both numerator and denominator by the factor 4; thus, $\frac{1}{3} = \frac{4}{12}$. In a similar manner $\frac{1}{4}$ is found to be $\frac{3}{12}$.

The *least common denominator* of several fractions is the least quantity which may be divided by each denominator without a remainder. Thus, 24 is the least common denominator of the fractions $\frac{5}{6}$, $\frac{3}{8}$, and $\frac{7}{12}$, and ab is the least common denominator of the fractions $\frac{1}{a}$ and $\frac{1}{b}$.

The least common denominator of several fractions contains all of the prime factors of the given denominators. Thus, either 96 or 192 is a common denominator of the fractions $\frac{5}{12}$ and $\frac{7}{16}$, and contains all their prime factors, 2, 2, 3, and 2, 2, 2, 2. The *least* common denominator, however, is neither 96 nor 192, but 48, because 48 is the least quantity which contains the prime factors of both 12 and 16 the greatest number of times that they appear in either one of these quantities.

$$12 = 2 \times 2 \times 3$$

$$16 = 2 \times 2 \times 2 \times 2$$

$$48 = 2 \times 2 \times 2 \times 2 \times 3$$

Since 2 is a common factor of 12 and 16 and appears four times in the latter, it is taken four times. There are no other common factors and therefore, the four 2's and the remaining factor 3, are all the factors which make up the least common denominator, and their product is 48.

Therefore, to find the least common denominator, separate the denominators into their prime factors and take each factor as many times as it appears in the denominator containing it the greatest number of times.

Examples. 1. Find the least common denominator of the fractions, $\frac{1}{3}$, $\frac{3}{4}$, $\frac{2}{9}$, and $\frac{7}{12}$.

SOLUTION. First, write the denominators in a row, as shown in the margin. Now the least quantity to contain 3, 4, 9, and 12 must be the smallest quantity that will contain the factors of each of them, but no other factors. Then all the prime factors that the common denominator contains must be found. 2 is a prime factor of 4 and 12. Therefore, it must be a factor of any quantity that contains 4 and 12 without a remainder. Divide the 4 and 12 by 2, writing the quotients below, carrying down the 3 and 9 which are not divisible by 2. Again it is seen that 2 is a factor of 2 and 6, and the operation is repeated, obtaining 3, 1, 9, and 3. Next dividing by 3, the result is 1, 1, 3, and 1. Now these final quotients have no common factor, and must be factors of the least common denominator just as 2, 2, and 3 are. Disregarding the 1's, $2 \times 2 \times 3 \times 3 = 36$. The result 36 is the least common denominator. Whenever 1's appear, they may be disregarded, as multiplying by the factor 1 produces no change in the quantity.

$$\begin{array}{r} 2)3; 4; 9; 12 \\ \hline 2)3; 2; 9; 6 \\ \hline 3)3; 1; 9; 3 \\ \hline 1; 1; 3; 1 \end{array}$$

2. Find the least common denominator of the fractions $\frac{5}{12ax}$, $\frac{7}{48by}$, and $\frac{5}{42bx}$.

SOLUTION. Dividing through by the successive factors, remainders are a , $4y$, and 7. Therefore, $x \times 2 \times 3 \times 2 \times b \times a \times 4y \times 7 = 336abxy$. The work can sometimes be shortened by dividing out any factor common to all the denominators even though it is not prime. It may be observed that 6 is such a common factor, and could have been used as a divisor in place of 2 and 3.

$$\begin{array}{r} x)12ax; 48by; 42bx \\ \hline 2)12a; 48by; 42b \\ \hline 3)6a; 24by; 21b \\ \hline 2)2a; 8by; 7b \\ \hline b)1a; 4by; 7b \\ \hline a; 4y; 7 \end{array}$$

Sometimes the least common denominator may be found by inspecting the fractions. In $\frac{1}{6}$, $\frac{5}{12}$, and $\frac{4}{7}$, since 12 is a multiple of 6, and any quantity which has 12 as a factor must contain 6, it is only necessary to find the least common denominator of the denominators 12 and 7, which is 84.

To reduce fractions to their least common denominator, first reduce each fraction to its lowest terms, and then find the least common

denominator of these fractions. Each fraction must now be changed to a fraction whose denominator is the least common denominator, which operation, it is readily seen, is merely the reducing of the fractions to higher terms. For example, in the preceding paragraph, the least common denominator of the fractions $\frac{1}{6}$, $\frac{5}{12}$, and $\frac{4}{7}$, is found to be 84; when the reduction is accomplished, $\frac{1}{6}$ becomes $\frac{14}{84}$; $\frac{5}{12}$ becomes $\frac{35}{84}$; and $\frac{4}{7}$ becomes $\frac{48}{84}$.

Examples. 1. Let $\frac{6}{18}$, $\frac{3}{4}$, $\frac{2}{9}$, and $\frac{7}{12}$, be reduced to equivalent fractions with a least common denominator.

SOLUTION. $\frac{1}{18}$ must be reduced to lower terms by dividing both numerator and denominator by 6, which changes the form to $\frac{1}{3}$. The least common denominator of $\frac{1}{3}$, $\frac{3}{4}$, $\frac{2}{9}$, and $\frac{7}{12}$ has just been found to be 36.

$$36 \div 3 = 12 \qquad 12 \times 1 = 12 \qquad \frac{1}{3} = \frac{12}{36}$$

$$36 \div 4 = 9 \qquad 9 \times 3 = 27 \qquad \frac{3}{4} = \frac{27}{36}$$

$$36 \div 9 = 4 \qquad 4 \times 2 = 8 \qquad \frac{2}{9} = \frac{8}{36}$$

$$36 \div 12 = 3 \qquad 3 \times 7 = 21 \qquad \frac{7}{12} = \frac{21}{36}$$

2. Reduce $\frac{15}{36ax}$, $\frac{7}{48by}$, and $\frac{5}{42bx}$ to equivalent fractions with a least common denominator.

SOLUTION. The fraction $\frac{15}{36ax}$ may be reduced by dividing both numerator and denominator by 3. $\frac{15}{36ax} = \frac{5}{12ax}$. The least common denominator of $\frac{5}{12ax}$, $\frac{7}{48by}$, and $\frac{5}{42bx}$ was found on Page 37 to be $336abxy$.

$$336abxy \div 12ax = 28by \qquad 28by \times 5 = 140by \qquad \frac{5}{12ax} = \frac{140by}{336abxy}$$

$$336abxy \div 48by = 7ax \qquad 7ax \times 7 = 49ax \qquad \frac{7}{48by} = \frac{49ax}{336abxy}$$

$$336abxy \div 42bx = 8ay \qquad 8ay \times 5 = 40ay \qquad \frac{5}{42bx} = \frac{40ay}{336abxy}$$

PROBLEMS FOR PRACTICE

1. Change $\frac{5}{6}$ and $\frac{3}{10}$ to fractions having a common denominator.
2. Change $\frac{4}{5}$, $\frac{7}{12}$, and $\frac{5}{6}$ to fractions having a common denominator.
3. Change $\frac{3}{7}$, $\frac{5}{8}$, $\frac{2}{3}$, and $\frac{1}{2}$ to fractions having a common denominator.

Change to proper or improper fractions using the least common denominator:

- | | |
|--|---|
| 4. $\frac{2}{36}$, $\frac{3}{16}$, $\frac{4}{10}$, and $\frac{4}{16}$. | 10. $\frac{3}{4}$, $\frac{2}{16}$, and $\frac{1}{4}$. |
| 5. $5\frac{1}{2}$, $2\frac{1}{4}$, and $1\frac{3}{8}$. | 11. $\frac{5}{b}$ and $\frac{b}{c}$ |
| 6. $\frac{4}{8}$, $\frac{7}{16}$, $3\frac{2}{3}$, and 7. | 12. $\frac{c}{ad}$, $\frac{b}{d}$, and $\frac{g}{de}$. |
| 7. $\frac{6}{8}$, $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{4}{7}$. | 13. $\frac{1}{m}$, $\frac{a}{n}$, $\frac{b}{mp}$, and $\frac{c}{np}$ |
| 8. $\frac{1}{2}$, $\frac{7}{16}$, and $\frac{4}{16}$. | 14. $\frac{a}{xy}$, $\frac{b}{xz}$, and $\frac{c}{yz}$. |
| 9. $\frac{2}{16}$, $\frac{1}{2}$, $\frac{3}{8}$, and $\frac{4}{8}$. | 15. $\frac{2a}{yz}$, $\frac{3b}{xy}$, and $\frac{4c}{y}$. |

ADDITION OF FRACTIONS

30. To add together two or more fractions, the fractional units must be of the same size; in other words, they must be reduced to a common denominator before the addition can be accomplished.

For example, suppose it is desired to add the fractions $\frac{1}{4}$ and $\frac{1}{3}$.

These may for convenience represent $\frac{1}{4}$ and $\frac{1}{3}$ of a foot. Now, it

is well known that a foot contains 12 inches, and the $\frac{1}{4}$ may be con-

sidered as 3 inches or $\frac{3}{12}$ of a foot, the $\frac{1}{3}$ as 4 inches, or $\frac{4}{12}$ of a foot

The required sum is then $3 + 4 = 7$ inches, or $\frac{3}{12} + \frac{4}{12} = \frac{7}{12}$ feet.

This process is graphically illustrated in Fig. 1.

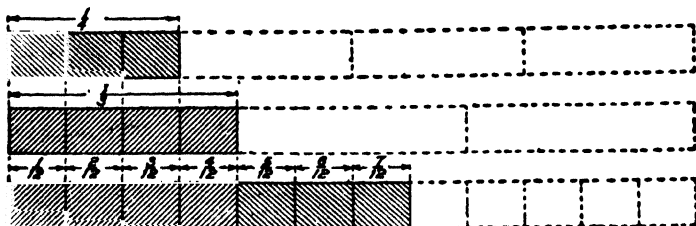


Fig. 1. Graphical Representation of Addition of Fractions

Now if an example be taken in which there are more than two fractions, it will be noted that the procedure is the same. $\frac{5}{8} + \frac{7}{12} + \frac{11}{24} + \frac{3}{4} = ?$ It may be seen by inspection or by the process already given that 24 is the least common denominator of the several fractions. The next step is to change the numerators of the fractions so that they will express the same value with the common denominator 24 as they now express with their respective denominators. $\frac{5}{8} = \frac{15}{24}$; $\frac{7}{12} = \frac{14}{24}$; $\frac{11}{24}$ will stand as written; $\frac{3}{4} = \frac{18}{24}$. Adding the several numerators and placing the sum over the common denominator the required sum is found to be $\frac{58}{24}$, which is an improper fraction. Reducing the improper fraction to a mixed quantity gives $2\frac{1}{2}$, or finally $2\frac{1}{2}$.

Again, to add $\frac{d}{2p}$, $\frac{d}{5p}$, and $\frac{d}{3p}$, first find the least common denominator, which is $30p$. Then $\frac{d}{2p} = \frac{15d}{30p}$; $\frac{d}{5p} = \frac{6d}{30p}$; $\frac{d}{3p} = \frac{10d}{30p}$; and $\frac{d}{2p} + \frac{d}{5p} + \frac{d}{3p} = \frac{15d}{30p} + \frac{6d}{30p} + \frac{10d}{30p} = \frac{31d}{30p}$.

If the problem involves letters only, the process is essentially the same. If $\frac{a}{n}$ and $\frac{b}{p}$ are to be added, they must first be reduced to

their common denominator, np . $\frac{a}{n} = \frac{ap}{np}$; $\frac{b}{p} = \frac{bn}{np}$; $\frac{a}{n} + \frac{b}{p} = \frac{ap}{np} + \frac{bn}{np}$. As the relation between ap and bn is not known, the sum of the numerators can only be represented by the expression $ap + bn$, giving the final sum $\frac{ap + bn}{np}$.

To add mixed quantities, add the entire quantities and the fractions separately, and if the fractional result of this addition is an improper fraction, reduce it to a mixed quantity and add the entire part to the sum of the entire quantities already obtained.

Examples. 1. $20\frac{2}{3} + 13\frac{1}{3} + 7\frac{1}{3} = ?$

SOLUTION. Adding the entire quantities together gives $20 + 13 + 7 = 40$. Reduce the fractions to higher terms having a common denominator and add, obtaining $\frac{16}{24} + \frac{12}{24} + \frac{3}{24} = \frac{31}{24} = 1\frac{7}{24}$. $40 + 1\frac{7}{24} = 41\frac{7}{24}$ as the final result.

$$\begin{array}{r} 20\frac{2}{3} = 20\frac{16}{24} \\ 13\frac{1}{3} = 13\frac{8}{24} \\ 7\frac{1}{3} = 7\frac{8}{24} \\ \hline 40\frac{31}{24} \\ = 40 + 1\frac{7}{24} = 41\frac{7}{24} \end{array}$$

2. Add $ab + \frac{b}{2}$ and $2ab + \frac{b}{3}$.

SOLUTION. The sum of the entire quantities is $3ab$. The least common denominator for $\frac{b}{2}$ and $\frac{b}{3}$ is 6, and the new forms of the fractions become $\frac{3b}{6}$ and $\frac{2b}{6}$ giving $\frac{5b}{6}$ as their sum. As $\frac{5b}{6}$ cannot be reduced to lower terms, the result is complete as shown in the margin.

$$\begin{array}{r} ab + \frac{b}{2} = ab + \frac{3b}{6} \\ 2ab + \frac{b}{3} = 2ab + \frac{2b}{6} \\ \hline 3ab + \frac{5b}{6} \end{array}$$

PROBLEMS FOR PRACTICE

Find the sum of

1. $\frac{5}{8}$, $1\frac{7}{8}$, and $\frac{1}{2}\frac{1}{4}$.

Ans. $1\frac{3}{4}$

2. $\frac{3}{10}$, $1\frac{7}{10}$, and $\frac{5}{15}$.

3. $6\frac{2}{3}$, $2\frac{3}{7}$, and $4\frac{4}{8}$.

Ans. $13\frac{2}{3}\frac{3}{7}$

4. $\frac{T}{2}$ and $\frac{T}{3}$.

5. $\frac{R}{N}$ and $\frac{M}{P}$.

Ans. $\frac{PR+MN}{NP}$

6. $d + \frac{4d}{6}$ and $4d + \frac{d}{2}$. Ans. $6d + \frac{d}{6}$

7. $a + \frac{c}{3}$ and $b + \frac{c}{4}$.

8. A room is $32\frac{3}{4}$ feet long and $29\frac{1}{4}$ feet wide. What is the distance around the room?

9. Three castings weigh respectively $225\frac{3}{8}$, $232\frac{5}{16}$, and $240\frac{1}{4}$ pounds. What is their total weight?

10. A steel rod is to be cut into five pieces; the first to be $4\frac{5}{8}$ inches long, the second $3\frac{3}{4}$ inches, the third $5\frac{1}{16}$ inches, the fourth $4\frac{5}{16}$ inches, and the fifth $1\frac{5}{8}$ inches. Find the length of the rod required.

11. A casting weighing $18\frac{3}{8}$ pounds has had $2\frac{1}{8}$ pounds of metal removed by the planer. How much did the original casting weigh?

SUBTRACTION OF FRACTIONS

31. *Fractions may be subtracted* only when they have a common denominator, and express quantities of like units.

Hence to *subtract proper fractions* reduce the given fractions to their equivalents, having a least common denominator, and write the difference of the numerators over the common denominator.

Examples. 1. Find the difference between $\frac{5}{6}$ and $\frac{8}{15}$.

SOLUTION. The fractions, when reduced to the least common denominator, become $\frac{25}{30}$ and $\frac{16}{30}$; their difference is, therefore, $\frac{9}{30}$, or reduced to lowest terms $\frac{3}{10}$.

$$\begin{aligned}\frac{5}{6} &= \frac{25}{30} \\ \frac{8}{15} &= \frac{16}{30} \\ \frac{5}{6} - \frac{8}{15} &= \frac{9}{30} = \frac{3}{10}\end{aligned}$$

2. Find the difference between $\frac{a}{p}$ and $\frac{a}{2p}$.

SOLUTION. The fraction $\frac{a}{p}$ may be reduced to $\frac{2a}{2p}$. Then, subtracting, $\frac{2a}{2p} - \frac{a}{2p} = \frac{2a}{2p} - \frac{1a}{2p} = \frac{1a}{2p}$ or $\frac{a}{2p}$.

$$\begin{aligned}\frac{a}{p} &= \frac{2a}{2p} \\ \frac{2a}{2p} - \frac{a}{2p} &= \frac{a}{2p}\end{aligned}$$

To *subtract mixed quantities*, subtract the fractional and entire parts separately, and add the remainders. If the mixed quantities are small, they may be reduced to improper fractions and subtracted.

Examples. 1. From $27\frac{5}{8}$ subtract $14\frac{1}{8}$.

SOLUTION. Subtracting the entire quantities, $27 - 14 = 13$. Reduce the fractions to higher terms having a common denominator and subtract, obtaining

$$\frac{20}{24} - \frac{15}{24} = \frac{5}{24}.$$

$$\begin{array}{r} 27\frac{5}{8} - 27\frac{1}{8} \\ 14\frac{1}{8} - 14\frac{1}{8} \\ \hline 13\frac{5}{8} \end{array}$$

2. From $21\frac{1}{4}$ subtract $1\frac{3}{7}$.

SOLUTION. The common denominator is 14, hence $2\frac{11}{14} - \frac{39}{14}$, and $1\frac{2}{7} - \frac{9}{7} = \frac{18}{14}$.

The difference is $\frac{21}{14}$, which may be reduced to the lowest terms by dividing both numerator and

$$\begin{array}{r} 21\frac{1}{4} - 1\frac{3}{7} \\ 1\frac{3}{7} - 1\frac{3}{7} \\ \hline \frac{21}{14} = \frac{3}{2} = 1\frac{1}{2} \end{array}$$

denominator by 7 giving $\frac{3}{2} = 1\frac{1}{2}$

To subtract a fraction or mixed quantity from an entire quantity, or from a mixed quantity in which the fraction of the minuend is less than the fraction of the subtrahend, one unit of the integer in the minuend must be written as a fraction. This is shown by the following example.

3. From 17 take $\frac{9}{11}$.

SOLUTION. First write one of the 17 units as a fraction having 11 parts.

Thus, $17 = 16 + \frac{11}{11}$. Now the subtraction may be accomplished without difficulty, giving $16\frac{2}{11}$.

$$\begin{array}{r} 17 = 16\frac{11}{11} \text{ minuend} \\ \quad \frac{9}{11} \text{ subtrahend} \\ \hline 16\frac{2}{11} \end{array}$$

In the case of letters, subtraction may be made after writing the entire quantity, as shown in the margin.

4. From m take $\frac{a}{p}$.

SOLUTION. The entire quantity m may be written in fractional form having the desired denominator by multiplying and dividing m by p , thus giving $\frac{mp}{p}$. The subtraction is then indicated as shown.

$$\begin{array}{r} m = \frac{mp}{p} \\ \frac{a}{p} \\ \hline \frac{mp - a}{p} \end{array}$$

PROBLEMS FOR PRACTICE

1. From $1\frac{2}{8}$ subtract $\frac{5}{8}$.

2. From $\frac{5}{7}$ subtract $\frac{3}{5}$.

3. From $1\frac{5}{8}$ subtract $2\frac{3}{4}$.

Ans. $\frac{4}{35}$

Ans. $1\frac{3}{8}$

4. From $1\frac{1}{2}$ subtract $\frac{5}{8}$

5. From $1\frac{2}{3}$ subtract $\frac{5}{12}$

Ans. $1\frac{1}{4}$

6. From $1\frac{7}{8}$ subtract $1\frac{1}{4}$

7. From $2\frac{5}{8}$ subtract $1\frac{1}{4}$

8. From $2\frac{1}{2}$ subtract $3\frac{3}{4}$

Ans. $1\frac{1}{4}$

9. A flagstaff $50\frac{1}{2}$ feet high was broken off in a storm so that $43\frac{7}{8}$ feet remained standing. How much was broken off?

10. A box and its contents weighed $75\frac{3}{8}$ pounds. The box alone weighed $31\frac{5}{8}$ pounds. What was the weight of its contents?

MULTIPLICATION OF FRACTIONS

32. *Multiplication of fractions* requires no such reduction of the fractions as was found necessary in addition and subtraction. Multiplying the numerator of a fraction multiplies the number of fractional units, their *size* remaining the same, and dividing the denominator multiplies the size of the fractional units, the *number* remaining the same. In many solutions cancellation will be found useful.

To multiply a fraction by an entire quantity, multiply the numerator of the fraction by the whole number, and write the product over the denominator, or divide the denominator of the fraction by the entire quantity, when it can be done without a remainder, and write the quotient as the denominator. Then reduce to lowest terms, and if necessary to a mixed quantity.

Examples. 1. Multiply $\frac{7}{8}$ by 4.

SOLUTION a. Multiplying the numerator by 4 increases the number of fractional units from 7 to 28, giving $\frac{28}{8}$, which, reduced to lower terms, becomes $\frac{7}{2}$ and finally as a mixed number $3\frac{1}{2}$.

$$\frac{7}{8} \times 4 = \frac{28}{8} = \frac{7}{2} = 3\frac{1}{2}$$

SOLUTION b. Dividing the denominator by the divisor is equivalent to cancellation. 4 divided into 8 is 2. The denominator is then 2, making the fractional unit 4 times as large. By the usual process $\frac{7}{2}$ is changed to $3\frac{1}{2}$.

$$\frac{7}{8} \times 4 = \frac{7}{2} = 3\frac{1}{2}$$

2. Multiply $\frac{7}{6p}$ by $3p$.

SOLUTION a. Multiply the numerator of the fraction by $3p$, and place over the denominator $6p$. $\frac{21p}{6p}$ may be reduced by dividing out p in the numerator and denominator.

$$\frac{7}{6p} \times 3p = \frac{7 \times 3p}{6p} = \frac{21p}{6p}$$

$$\frac{21p}{6p} = \frac{21}{6} = 3\frac{3}{6} = 3\frac{1}{2}$$

SOLUTION b. The same result would have been obtained more briefly by dividing the denominator of the fraction by $3p$. By cancellation the result $\frac{7}{2}$ is immediately obtained.

$$\frac{7}{\cancel{6p}^2} \times \cancel{3p}^1 = \frac{7}{2}$$

$$\frac{7}{2} = 3\frac{1}{2}$$

To multiply one fraction by another, multiply together the two numerators and place this product over the product of the two denominators, or indicate the multiplication, and cancel wherever possible.

3. Multiply $\frac{5}{8}$ by $\frac{2}{5}$.

SOLUTION a. The 2 of the multiplier makes the product larger, and therefore, the numerator of the multiplicand is multiplied by 2. The 5 of the multiplier makes the product smaller, and hence, the denominator is multiplied by 5.

$$\frac{5}{8} \times \frac{2}{5} = \frac{5 \times 2}{8 \times 5} = \frac{10}{40} = \frac{1}{4}$$

SOLUTION b. The solution in the margin shows how the multiplication may be more briefly performed by cancellation.

$$\frac{\cancel{5}^1}{\cancel{8}_4} \times \frac{\cancel{2}_2}{\cancel{5}_1} = \frac{1}{4}$$

4. Multiply $\frac{d}{4}$ by $\frac{2a}{b}$.

SOLUTION. By the usual method the numerator is equal to the product of d and $2a$, or $2ad$, and the denominator is $4b$.

$$\frac{d}{4} \times \frac{2a}{b} = \frac{d \times 2a}{4 \times b} = \frac{2ad}{4b}$$

$$\frac{2ad}{4b} = \frac{ad}{2b}$$

To multiply mixed quantities, change them to improper fractions and multiply as before.

5. Multiply $2\frac{2}{7}$ by $5\frac{1}{4}$.

SOLUTION. Write these as improper fractions. $\frac{16}{7}$ and $\frac{21}{4}$ is obtained. Cancellation gives 12 at once.

$$\frac{\cancel{16}^4}{\cancel{7}_1} \times \frac{\cancel{21}^3}{\cancel{4}_1} = 12$$

To multiply mixed quantities and whole numbers or fractions, change the mixed quantities and whole numbers to improper fractions and multiply.

6. Find the product of $2\frac{2}{7}$, $2\frac{5}{8}$, $\frac{7}{3p}$, and $6p$.

SOLUTION:
$$\frac{16}{7} \times \frac{21}{8} \times \frac{7}{3p} \times \frac{6p}{1} = 84$$

PROBLEMS FOR PRACTICE

Find the product of

1. $\frac{5}{8}$ and $\frac{4}{7}$.

Ans. $\frac{5}{14}$

2. $\frac{400}{9}$ and $\frac{360}{17}$.

3. $\frac{120}{17}$ and $\frac{10}{15}$.

4. $12\frac{3}{4}$ and $11\frac{5}{9}$.

5. $11\frac{2}{7}$ and $8\frac{4}{5}$.

Ans. $99\frac{1}{5}$

6. $12\frac{3}{8}$ and 7.

7. $6\frac{1}{6}$; $\frac{4}{6}$; and $4\frac{1}{6}$.

8. $\frac{2}{3}$; $\frac{4}{3}$; $1\frac{8}{3}$; and $1\frac{2}{1}$.

9. A water-tube boiler has a grate surface of $27\frac{3}{10}$ square feet. It burns $15\frac{1}{2}$ pounds of coal per square foot per hour. How much does it burn in $1\frac{1}{2}$ hours?

10. What will $14\frac{1}{2}$ yards of curbing cost at $\$6\frac{1}{2}$ per yd.?

11. What will $17\frac{1}{8}$ tons of coal cost at $\$4\frac{7}{10}$ per ton?

12. A point on the fly wheel of an engine travels $16\frac{2}{3}$ feet per revolution. How far does it travel in $29\frac{5}{6}$ revolutions? How long would it take if the wheel made one revolution in $2\frac{3}{4}$ seconds?

DIVISION OF FRACTIONS

33. *Division of fractions* is the reverse of multiplication. Dividing the numerator of a fraction divides the number of fractional units, their size remaining the same, and multiplying the denominator also divides the size of the fractional units, the number remaining the same. This latter process is equivalent to inverting the divisor and multiplying. Thus $\frac{7}{8} \div \frac{3}{4}$ and $\frac{7}{8} \times \frac{4}{3}$ are

equivalent expressions. The fraction after inverting is called the *reciprocal* of the original fraction.

To divide a fraction by an entire quantity, change the entire quantity to a fractional form with one as a denominator; invert the fractional form and multiply; or, divide the numerator of the fraction by the entire quantity.

Examples. 1. Divide $\frac{8}{9}$ by 4.

SOLUTION. The reciprocal of 4 is $\frac{1}{4}$.
This multiplied by $\frac{8}{9}$, using, cancellation gives $\frac{8}{9} \div 4 = \frac{8}{9} \times \frac{1}{\cancel{4}^1} = \frac{2}{9}$
the quotient $\frac{2}{9}$.

2. Divide $\frac{a}{b}$ by a .

SOLUTION. The reciprocal of a is multiplied by the fraction, and by cancellation the result is obtained.

$$\frac{a}{b} \div a = \frac{a}{b} \times \frac{1}{\cancel{a}^1} = \frac{1}{b}$$

3. Divide 13 by $\frac{3}{7}$.

SOLUTION. Invert the fraction and multiply. Multiplying 13 by the reciprocal of $\frac{3}{7}$ gives at once $\frac{91}{3}$ or $30\frac{1}{3}$.

$$13 \div \frac{3}{7} = 13 \times \frac{7}{3} = \frac{91}{3} = 30\frac{1}{3}$$

4. Divide d by $\frac{d}{c}$.

SOLUTION. The reciprocal of the fraction is multiplied by d , and cancellation is used.

$$d \div \frac{d}{c} = \frac{\cancel{d}^1 \times c}{\cancel{d}_1} = c$$

To divide a fraction by a fraction, invert the divisor, that is, the second fraction, and then proceed as in multiplication of fractions.

5. Divide $\frac{3}{4}$ by $\frac{7}{8}$.

SOLUTION. The multiplication is accomplished by multiplying $\frac{3}{4}$ by the reciprocal of $\frac{7}{8}$, using cancellation. The result $\frac{6}{7}$ is obtained at once.

$$\frac{3}{4} \div \frac{7}{8} = \frac{3}{\cancel{4}^1} \times \frac{\cancel{8}^2}{7} = \frac{6}{7}$$

6. Divide $\frac{d}{p}$ by $\frac{a}{c}$.

SOLUTION. The process is exactly the same as used in the preceding examples.

$$\frac{d}{p} \div \frac{a}{c} = \frac{d}{p} \times \frac{c}{a} = \frac{cd}{ap}$$

When division involves a mixed number, the mixed number may be reduced to an improper fraction and one of the preceding methods used.

7. Divide $2\frac{1}{2}$ by $1\frac{1}{3}$.

SOLUTION.

$$2\frac{1}{2} \div 1\frac{1}{3} = \frac{5}{2} \div \frac{4}{3} = \frac{5}{2} \times \frac{3}{4} = \frac{15}{8} = 1\frac{7}{8}$$

When a fraction takes the form called a *complex fraction*, that is, having a fraction in numerator or denominator or both, as $\frac{\frac{7}{8}}{4}$, the division is handled as in the examples just discussed. That is, $\frac{7}{8} \div 4 = \frac{7}{8} \times \frac{1}{4} = \frac{7}{8 \times 4} = \frac{7}{32}$.

8. Divide $1\frac{1}{2}$ by $\frac{3}{4}$.

SOLUTION. The complex fraction is first reduced to a simple fraction; the division by the second fraction is then indicated, and this second fraction inverted so as to allow multiplication. Cancellation reduces the fractions to the final result.

$$\begin{aligned} \frac{9}{11} \div \frac{2}{5} &= \frac{9}{11} \times \frac{5}{2} \\ \left(\frac{9}{11} \times \frac{5}{2} \right) \div \frac{3}{4} &= \frac{9}{11} \times \frac{5}{2} \times \frac{4}{3} \\ &= \frac{9}{11} \times \frac{5}{2} \times \frac{2}{3} = \frac{30}{11} = 2\frac{8}{11} \end{aligned}$$

PROBLEMS FOR PRACTICE

Divide:

1. $\frac{7}{8}$ by $\frac{4}{5}$. Ans. $1\frac{1}{4}$

2. $\frac{7}{8}$ by $\frac{1}{4}$.

3. $1\frac{2}{3}$ by $\frac{3}{4}$.

4. $\frac{1}{2}$ by $\frac{7}{8}$.

5. $7\frac{1}{2}$ by $4\frac{1}{2}$. Ans. $1\frac{2}{3}$

6. A railroad $16\frac{1}{2}$ miles long cost \$66,937; find cost per mile.

7. A steam pump delivers $2\frac{1}{2}$ gallons per stroke. It delivers

6. $\frac{ft}{s}$ by $\frac{f}{b}$. Ans. $\frac{bt}{s}$

7. $\frac{mt}{a}$ by $\frac{b}{a}$.

330 gallons in $2\frac{1}{2}$ minutes. How many strokes does it make per minute? SUGGESTION: First find gallons delivered per minute.

DECIMALS

34. Decimal Fractions. It has been shown that common fractions may be reduced to higher or lower terms, *i. e.*, to fractions having any desired denominator. For example, the fraction $\frac{3}{4}$ may be reduced to one having a denominator of 12 by multiplying both terms by a number which will change the denominator to 12, *viz*, by 3. Thus $\frac{3}{4} \times \frac{3}{3} = \frac{9}{12}$, which is the desired fraction. The same fraction may have the denominators, 20, 32, 40, 60, and 80 by multiplying both terms successively by 5, 8, 10, 15, and 20, giving the fractions $\frac{15}{20}$, $\frac{24}{32}$, $\frac{30}{40}$, $\frac{45}{60}$, and $\frac{60}{80}$; all of these fractions are of *equal value* with the original fraction.

Now, suppose it is desirable to reduce all fractions to a *standard denominator* of 10, 100, 1,000, etc. Under this system the fraction $\frac{3}{4}$ will have successively the forms $\frac{7\frac{1}{2}}{10}$, $\frac{75}{100}$,

$\frac{750}{1000}$, etc., while another fraction like $\frac{7}{8}$ will have the forms $\frac{8\frac{1}{2}}{10}$, $\frac{87\frac{1}{2}}{100}$, and $\frac{875}{1000}$. Notice that the first form for $\frac{3}{4}$ and the first

and second for $\frac{7}{8}$ have fractions in the numerators. This makes an awkward combination which may always be avoided in dealing with such fractions by using the higher forms which contain no fraction in the numerator, as $\frac{75}{100}$ and $\frac{875}{1000}$.

However, it will be evident that if all fractions are so reduced, many of them will have fractional numerators, even though the denominators are made higher powers of 10, and hence, in cutting off the extra fractions certain errors will be made. It will be learned in the section on Per Cent that such errors, if they are small, are usually neglected. The fraction $\frac{1}{13}$, for example, may be given

a denominator of 10,000, making a fraction $\frac{769}{10000}$, which may be considered equal to $\frac{769}{10000}$ or even $\frac{77}{1000}$ without any appreciable difference.

Such fractions, as $\frac{75}{100}$, $\frac{875}{1000}$, and $\frac{77}{1000}$, etc., having 10, 100, 1000, etc., as denominators are called *decimal fractions*. In order to simplify this system and thus avoid writing the denominator, decimal fractions are expressed in another form which consists in writing only the numerator and placing a point (.) so that the number of places on the right of it shall be equal to the number of ciphers in the denominator. The decimal fractions given above, when expressed in this way, become .75, .875, .077, and when so used they are called *decimals*.

The point (.) is called the *decimal point* and its office is to mark the beginning of the decimal or separate it from a whole number.

A pure decimal has only decimal places, as, .93678.

A mixed decimal consists of a whole number and a decimal, as 364.23.

The *decimal* system is merely an extension of the ordinary number system to the *right* of units place, with a decimal point to indicate the boundary line. This is clearly shown in Table V.

TABLE V

Thousands	Hundreds	Tens	Units	Decimal Point	Tenths	Hundredths	Thousandths	Ten-Thousandths	Hundred-Thousandths	Millionths	Ten-Millionths	Hundred-Millionths
6	4	3	9	.	3	6	5	7	9	8	2	3

In reading the decimals, use the names just given to represent the places, omitting the word *and* except at the decimal point. The

following examples give the proper reading and the corresponding figures for a few cases:

1. Nine tenths 0.9
2. Ninety-five hundredths.....0.95
3. Nine hundred fifty-four thousandths.....0.954
4. Six thousand one ten-thousandths.....0.6001
5. Six and one ten-thousandth.....6.0001

The number shown in Table V is read thus: Six thousand, four hundred thirty-nine, and thirty-six million, five hundred seventy-nine thousand, eight hundred twenty-three hundred-millionths.

The United States and some other countries use the decimal system in money. For instance, take the sum of \$13.74. This may be considered to be made up of 1 ten dollar bill (tens position), 3 one dollar bills (units place, dollars being considered units), 7 dimes or tenths of a dollar (tenths position), and 4 cents or hundredths of a dollar (hundredths position). It is read thirteen dollars and seventy-four cents, and means thirteen dollars and seventy-four hundredths of a dollar. It is, of course, better to say seventy-four hundredths than seven tenths plus four hundredths, just as it is better to say seventy-four cents than to say seven dimes and four cents.

PROBLEMS FOR PRACTICE

Write in words:

- | | |
|-----------|------------|
| 1. .965 | 4. .10792 |
| 2. 3.8506 | 5. .010952 |
| 3. 5.0061 | 6. .4563 |

Write in figures:

7. Seven thousand eight hundred forty-nine hundred-thousandths.
8. Sixteen thousandths.
9. One hundred thirteen ten-thousandths.
10. Six hundred and thirty-three thousand seven hundred fifty-eight millionths. Ans. 606.033758
11. Twenty-nine hundredths.
12. Twelve and twenty-seven hundredths.
13. Four hundred seventy-two and four hundred eighty-seven ten-thousandths.

ADDITION OF DECIMALS

35. In *addition of decimals*, place the numbers so that the decimal points will be directly under each other, regardless of the number of figures. Having done this, add the figures as in addition of whole numbers. The last step is to place a decimal point in the sum directly under the column of decimal points.

Example. Add 1.75; 62.625; and 3.937.

SOLUTION. Following the process given above and placing the decimal point in the sum directly under the column of decimal points, 68.312 is obtained.

$$\begin{array}{r} 1.75 \\ 62.625 \\ 3.937 \\ \hline 68.312 \end{array}$$

The greatest accuracy must be exercised in using decimals. A decimal point is more important than any figure, because a misplaced decimal point increases the error at least ten times.

SUBTRACTION OF DECIMALS

36. In *subtraction of decimals*, the process is exactly the same as in the case of whole numbers.

Place the subtrahend under the minuend with the decimal points directly under each other. Subtract as in whole numbers. In the remainder, place the decimal point in the same column and directly under the other decimal points.

Examples. 1. From 5.17 subtract .01.

SOLUTION. Subtract as in whole numbers paying careful attention to the decimal point.

$$\begin{array}{r} 5.17 \\ .01 \\ \hline 5.16 \end{array}$$

2. From 128 subtract 96.307.

SOLUTION. This is an example in which the minuend is a whole number, which necessitates placing a decimal point to the right of the whole number, and annexing ciphers. In doing this the integer is changed to a mixed decimal. Adding ciphers to the right of the decimal point multiplies both the numerator and denominator of the decimal fraction by the same number, which, although changing its form, does not change its value. Carrying out the subtraction as in whole numbers, the remainder 31.693 is obtained.

$$\begin{array}{r} 128.000 \\ 96.307 \\ \hline 31.693 \end{array}$$

3. From 134.089 subtract 93.

SOLUTION. Placing a decimal point at the right of 93, subtract as before.

$$\begin{array}{r} 134.089 \\ 93. \\ \hline 41.089 \end{array}$$

PROBLEMS FOR PRACTICE

1. From 25.38 take 14.05
2. From 39.85 take 29.755 Ans. 10.095
3. From 72.189 take 35.976 Ans. 36.213
4. From 78.896 take 53.5987
5. From 21.12 take 12.31
6. From 6.325 take 1.0345
7. From 6.45 take 2.3375
8. From 81.35 take 11.679 Ans. 69.671
9. From a cistern that contained 30.5 barrels of water, 25.75 barrels were drawn off. How much water remained in the cistern?
10. A hundred pounds of coke were found to contain 5.79 pounds of ash and .597 pounds of sulphur. The rest was combined carbon. How much combined carbon was there?
11. In 1 pound of brazing metal there are .5 pound of copper and .125 pound of tin. The remainder is zinc. How much zinc is there?
12. An iron casting weighed in the rough 22.75 pounds, and when finished, it weighed only 16.875 pounds. How much had been taken off in the process?

MULTIPLICATION OF DECIMALS

37. In *multiplying decimals* proceed as with whole numbers, paying no attention to the decimal point until all the figures in the product are obtained. Then point off as many places in this product as the total number of places in both multiplicand and multiplier, prefixing ciphers if necessary.

Examples. 1. Multiply .397 by 41.

SOLUTION. Since there are three decimal places in the multiplicand and none in the multiplier, point off 3 + 0 places in the product.

$$\begin{array}{r}
 .397 \\
 41 \\
 \hline
 397 \\
 1588 \\
 \hline
 16.277
 \end{array}$$

2. Multiply .027 by .05.

SOLUTION. First write the decimals so that the multiplication may be most readily performed. 27 is multiplied by 5 as if both were integers. Ciphers are then prefixed to 135 until the product contains 3 + 2 = 5 decimal places.

$$\begin{array}{r}
 .027 \\
 .05 \\
 \hline
 .00135
 \end{array}$$

To multiply a decimal or a mixed number expressed as a decimal by 10, 100, 1,000, etc., move the decimal point as many places to the right as there are ciphers in the multiplier. If there are not figures enough for this, annex sufficient ciphers.

Observe the following:

$$.046 \times 1 = .046$$

$$.046 \times 100 = 4.6$$

$$.046 \times 10 = .46$$

$$.046 \times 10000 = 460.$$

In the last two multiplications the product contains whole numbers. The reason for this is that after the decimal point has been moved the required number of places to the right, the cipher which comes before is dropped since it has no effect on the value. In the last case it is necessary to annex a cipher to the 46 to give the number of places required by the four ciphers in the multiplier.

After a multiplication is completed, if ciphers occur at the right of the decimal point with no figures following, the ciphers may be dropped. Thus $12.4 \times 10.5 = 130.20$. The result should, therefore, be written 130.2.

PROBLEMS FOR PRACTICE

Find the product of

1. $.876$ and $.375$

Ans. $.3285$

2. 72.2 and $.055$

3. 3.62 and $.0037$

4. 15.8 and $.0855$

5. 2.53 and $.00635$

Ans. $.0160655$

6. $.765$ and $.067$

7. 18.46 and 1.007

Ans. 18.58922

8. $.00076$ and $.0015$

9. Thirty-four million and twenty-six millionths.

10. Eight hundred and forty-two thousandths and five hundred thousand.

11. A U. S. gallon of water weighs 8.335 pounds. What is the weight of 17.3 gallons?

Ans. 144.1955 lbs.

12. A steamship sailed at an average speed of 325.75 miles per day. If another steamer sailed from the same port at the same time and in the same direction at the rate of 395.25 miles per day, how far apart were they in 5.5 days?

Ans. 382.25 miles.

13. A cubic foot of water weighs 62.3 pounds. How much will 177.3 cubic feet of water weigh? Ans. 11,045.79 lbs.

14. A steam pump delivers 26.4 gallons per stroke. A gallon weighs 8.335 pounds. What weight of water will be delivered in 117 strokes?

15. In one pound of phosphor bronze, .925 is copper, .07 is tin, and .005 is phosphorus. How much of each (copper, tin, and phosphorus) is there in 369.5 pounds?

16. A round bar of rolled iron $2\frac{1}{8}$ inches in diameter weighs 11.1 pounds per foot. What is the weight of a bar of the same diameter and material which is 9.33 feet long?

17. A train made an average speed of 1.33 miles per minute. How many miles did it cover in 17.5 minutes?

DIVISION OF DECIMALS

38. In division of decimals proceed as with whole numbers paying no attention to the decimal point until the quotient is obtained. Then point off in the quotient as many decimal places as those in the dividend exceed those in the divisor. This, it will be noted, is the reverse of the process just given in multiplication of decimals.

It should be remembered that *while the dividend may contain more decimal places than the divisor, it must contain at least as many.* To bring this about, annex as many ciphers as necessary to the right of the decimal point in the dividend.

When the division does not come out evenly, annex ciphers to the dividend and continue the division so as to give at least two decimal places in the quotient.

Examples. 1. Divide 36.744 by 24.

SOLUTION. No attention is paid to the decimal point until all the figures of the quotient are obtained. As there are no decimal places in the divisor and three in the dividend, the number of decimal places in the quotient is three.

$$\begin{array}{r}
 1.531 \\
 24 \overline{) 36.744} \\
 \underline{24} \\
 127 \\
 \underline{120} \\
 74 \\
 \underline{72} \\
 24 \\
 \underline{24} \\
 0
 \end{array}$$

2. Divide .196 by .004.

SOLUTION. There are three decimal places in the divisor and three in the dividend. Therefore, none will be found in the quotient. The decimal point is placed to the right of the result.

$$\begin{array}{r} .004 \overline{) .196} \\ 49. \end{array}$$

3. Divide .0027 by 1.35.

SOLUTION. It is seen that 135 is not contained in 27. Therefore, annex a cipher to the dividend making 270, and the divisor 135 is contained twice without a remainder. Since a cipher was added to the dividend, it contains five decimal places, the divisor contains two, and hence, in the quotient there must be five minus two, or three decimal places. In order to have three decimal places, two ciphers must be prefixed to the quotient, and the decimal point placed before them.

$$\begin{array}{r} .002 \\ 1.35 \overline{) .00270} \\ \underline{270} \\ 0 \end{array}$$

4. Divide 7 by 8.

SOLUTION. As 8 is not contained in 7 an integral number of times it is necessary to annex ciphers, the decimal point being placed to the right of 7. The division is then carried out as with whole numbers and the number of decimal places in the quotient is equal to the number in the dividend.

$$\begin{array}{r} 8 \overline{) 7.000} \\ \underline{.875} \end{array}$$

By performing the division which the fraction only expresses, a *common fraction* becomes a *decimal*. Thus, $\frac{3}{8}$, $\frac{5}{24}$, $\frac{7}{16}$, and $\frac{15}{32}$ may be expressed as decimals by dividing 3 by 8, 5 by 24, 7 by 16, and 15 by 32, giving respectively .375, .208 + *, .437 +, and .469 -.

PROBLEMS FOR PRACTICE

Divide:

- | | |
|---------------------|---------------|
| 1. 183.375 by 489 | Ans. .375 |
| 2. 67.8632 by 32.8 | |
| 3. 67.56785 by .035 | |
| 4. .567891 by 8.2 | Ans. .06925 + |
| 5. .1728 by 100 | Ans. .001728 |
| 6. 13.50192 by 1.38 | |
| 7. 783.5 by 6.25 | |
| 8. 983 by 6.6 | Ans. 148.93 + |

* The + and - signs after the decimals indicate that the true values are slightly more or less than the values given.

9. 3 by 8 Ans. .375
10. 1.95 by .45
11. How much gas at \$1.25 per thousand cubic feet can be bought for \$17.50?
12. The distance between two places is 167.75 miles. How long will it take a steamer to run the distance if she makes, on the average, 12.5 miles per hour?
13. If a freight train runs at the rate of 15.75 miles per hour, how long will it take it to run 189 miles?
14. A lot of 22,840 railroad ties cost \$39,867.22. What was the cost per hundred?
15. Seven readings of a dynamometer gave the following horse-powers: 17.31, 17.95, 18.13, 17.79, 17.87, 17.63, and 17.47. What was the average reading?
- Reduce to decimals:
16. $\frac{5}{8}$ 18. $\frac{32}{64}$
17. $\frac{3}{4}$ 19. $\frac{5}{16}$
20. Change .756 to the nearest 12th. Ans. $\frac{9}{12}$
21. Change .564 to the nearest 64th.

PERCENTAGE

39. The idea of a fraction of anything, whether it be a common fraction like $\frac{1}{15}$ or a decimal fraction like .001, should now be quite clear. The fraction $\frac{1}{15}$ merely serves to show that something has been divided into 15 parts and one of these parts has been taken. If $\frac{1}{1000}$ of the same thing were taken, this portion would evidently be smaller than the $\frac{1}{15}$. Again, a given quantity is a bigger part of a small group than it is of a large group. The results of one day's excavating may be $\frac{1}{30}$ of all the work to be done on one section of a tunnel, but only $\frac{1}{3000}$ of the total work if there are 100 sections.

The coin which a man with a bank account gives to a boy in exchange for a small service is an insignificant part of the man's money, but becomes at once in the boy's hands a large and important portion of his capital.

Evidently, therefore, there must be some standard number by which to determine the importance of quantities which are to be measured or compared. *This standard group number is 100* and all changes in numbers can be so reduced as to be expressed in terms of 100 parts; when so reduced the result is called the *per cent* of change in that number (written % and meaning *by the hundred*). For example, a change of 2 parts in 200, that is, a change from 200 to 198, may be reduced by dividing by 2, to equal 1 part in 100, or 1%; a change of 10 parts in 500 when divided by 5 is found to be the same as 2 parts in 100, or 2%. A man who has one hundred \$1 bills and spends one of them, has decreased his capital by 1 part in 100 or 1%; if he spends \$3, \$5, or \$15, his capital has been diminished by 3%, 5%, and 15%, respectively. Again, a man who buys corn at 33c and sells at 35c makes 2c on every 33c, 4c on 66c, or 6c on 99c, say \$1; that is, his gain is 6%.

Fractions may be expressed in terms of per cent. $\frac{1}{15}$ may be expressed as 1 part in 15 or $6\frac{2}{3}$ parts in 100, roughly 7%. This means that $\frac{1}{15}$ of any thing, as the weight of a casting, is about 7% of it. The fraction $\frac{1}{1000}$ means 1 part in 1000, or .1 part in 100, or .1%. Similarly $\frac{1}{8} = 12\frac{1}{2}\%$; $\frac{1}{3} = 33\frac{1}{3}\%$; $\frac{1}{10} = 10\%$, etc.

Examples. 1. What per cent of error is allowed in a shop if a steel shaft 2 inches in diameter must be true to the third decimal place?

SOLUTION. Accuracy to the third decimal place means that the shaft can have a diameter of 2.001 inches or 1.999 inches. This is 1 part in 2000, .5 part in 1000, or .05 part in 100. The permissible error is, therefore, .05%.

2. A base line 500 feet long was measured by a party of surveyors. The total error in this measurement proved to be 2 inches. What is the per cent of error?

SOLUTION. 500 feet = 500×12 inches = 6000 inches. An error of 2 inches is, therefore, an error of 2 parts in 6000, or 1 part in 3000. This is $\frac{1}{3000}$ of a part in 1000 or $\frac{1}{300}$ in 100—that is, $\frac{1}{300}$ of 1% or approximately .03%.

3. A contractor figured the cost of a certain piece of work at \$6750; he added 10% for delays and accidents and 20% for profit. What was his profit and what was the amount of his bid?

SOLUTION. The cost price of the work is \$6750. 10% added for accidents is $6750 \times .10$ or \$675. 20% added for profit will be $2 \times 675 = \$1350$. The amount of his bid should be, therefore, $\$6750 + \$675 + \$1350 = \$8775$.

4. What error is made in using the reducing factor 2.5 centimeters to the inch instead of the actual value 2.540 centimeters?

SOLUTION. $2.540 - 2.5 = .04$, the error. This is an error of 4 hundredths in 254 hundredths or 4 parts in 254 or about 1 in 64. Increasing 64 by $\frac{1}{4} = 96$, approximately 100, and increasing 1 by $\frac{1}{4} = 1\frac{1}{4}$. Therefore, 1 part in 64 equals $1\frac{1}{4}$ parts in 100 or $1\frac{1}{4}\%$. The error is $1\frac{1}{4}\%$.

To put the matter in rule form it is necessary to give names to the different quantities as follows:

The quantity of which the per cent is taken is called the *base*.

The number of hundredths or % of the base to be taken is called the *rate*.

The result obtained by taking the required per cent of the base is called the *percentage*.

Rules. (a) *The product of the rate and the base equals the percentage.*

(b) *The percentage divided by the base equals the rate.*

(c) *The percentage divided by the rate equals the base.*

(d) *To change a number to the per cent form multiply by 100 and annex the sign %.*

(e) *To change a number indicating per cent back to the original figures, drop the % sign and divide by 100. For example $\frac{1}{4} = .25$ and this expressed as a per cent equals $.25 \times 100$ or 25%. Conversely $20\% = \frac{20}{100} = \frac{1}{5} = .2$.*

These rules will help in many cases, but for applications to shop work or calculations in general, the more informal method given above is strongly recommended.

Examples. 1. Find the per cent of error made by a machinist who took a dimension from a drawing as 4.72 inches and finished his piece with a dimension of 4.79 inches.

SOLUTION a. The difference between the two dimensions is .07 and as there are approximately 470 hundredths in the dimension, the error is 7 parts in 470, about 1 in 65, or 1.5 in 100, i. e., 1.5% error. Ans.

SOLUTION b. The base is 4.72, the percentage is .07, to find the rate.
 $.07 \div 4.72 = .0148$ or 1.48 + %

NOTE. In solution *a*, approximate values are taken so as to arrive at the conclusion without labored calculation. In this way it is easy to calculate % of error mentally with sufficient accuracy. In solution *b* it is necessary to move the decimal point of the quotient two places to the right in order to obtain the %.

2. In a town of 8,000 inhabitants, during an epidemic of typhoid fever, there were 114 deaths from the disease. Find the per cent of deaths.

SOLUTION. 114 deaths in every 8,000 equals about 14 in 1,000 or 1.4 in 100, i. e., 1.4% Ans.

3. A business firm has a factory stock valued at \$28,000. At inventory time the firm allows a depreciation of 8%. What is the loss?

SOLUTION a. 8% depreciation means \$8 loss on every \$100, and as there are 280 of these \$100, the loss is $\$8 \times 280 = \$2240$. Ans.

SOLUTION b. The base is \$28,000, the $\$28000 \times .08 = \$2240$. Ans.
 rate 8%. The percentage will be the loss.

NOTE. In such problems as No. 3, solution *b* is the better method.

4. The total weight of a freight car when loaded is 148,600 pounds, and the weight of the empty car is 41,700 pounds; what per cent of the entire weight is the weight of the empty car? What per cent of the entire weight is the freight carried?

SOLUTION. $41,700 \div 148,600 = .28$ or 28 + % Ans.

Evidently the weight of the freight is (100 - 28) or 72% of the total weight.

The method of *per cent* may be used also in determining how far to carry the calculations in various problems. The multiplication of the two numbers 4.75 and 65.4 gives 310.65. If it is asked whether it is necessary to carry the second decimal place (.05) in this product, this could be determined at once by finding what per cent .05 is of the whole number. .05 in 310.65 is roughly 5 in 31,000 or 1 in 6,000, which is $\frac{1}{60}$ in 100 or $\frac{1}{60}\%$, a very small error. The slide rule, by which engineers often make important calculations, does not give an accuracy of more than $\frac{1}{2}\%$ to $\frac{1}{10}\%$. Consequently in all

of the problem calculations, the carrying of the results to four or five numerical places (this means the actual figures and does not count the ciphers) is usually sufficient. It is with this probable error in view that reports on elections, census, etc., are given in *round numbers*. The population of the U. S. in 1900 may have been 84,755,643, but for all practical purposes 85 million is sufficient. A contractor might figure that a piece of work would be worth \$12,743.22 but his bid probably would be \$12,750.

PROBLEMS FOR PRACTICE

Express in two other ways:

1. $\frac{1}{4}, \frac{1}{8}, \frac{1}{25}, \frac{3}{8}$. 3. 3%, 5%, 70%. 5. 20%, $66\frac{2}{3}\%$, 125%.

2. .25, $.12\frac{1}{2}$, .75. 4. $\frac{1}{12}, \frac{1}{10}, \frac{1}{20}$. 6. $.06\frac{1}{2}$, $62\frac{1}{2}$, $.33\frac{1}{3}$.

7. What per cent is \$25 of \$75? \$107.03 of \$1,046? \$7,894 of \$11,841? Ans. $33\frac{1}{3}\%$; $51\frac{1}{2}\%$; $66\frac{2}{3}\%$.

8. A merchant lost \$3,000 of his capital and had \$21,000 remaining. What per cent of his capital did he lose? Ans. $12\frac{1}{2}\%$.

9. The value of the ratio of the circumference to the diameter of a circle (usually designated by the Greek letter π , pi) is 3.1416, with an approximate value of $\frac{22}{7}$. What per cent of error is made

in this approximation? Ans. $.04\%$

10. A steam engine furnishes 350 horse-power to a dynamo, which transforms this into electrical energy with a loss of 8%. Find the horse-power supplied by the dynamo.

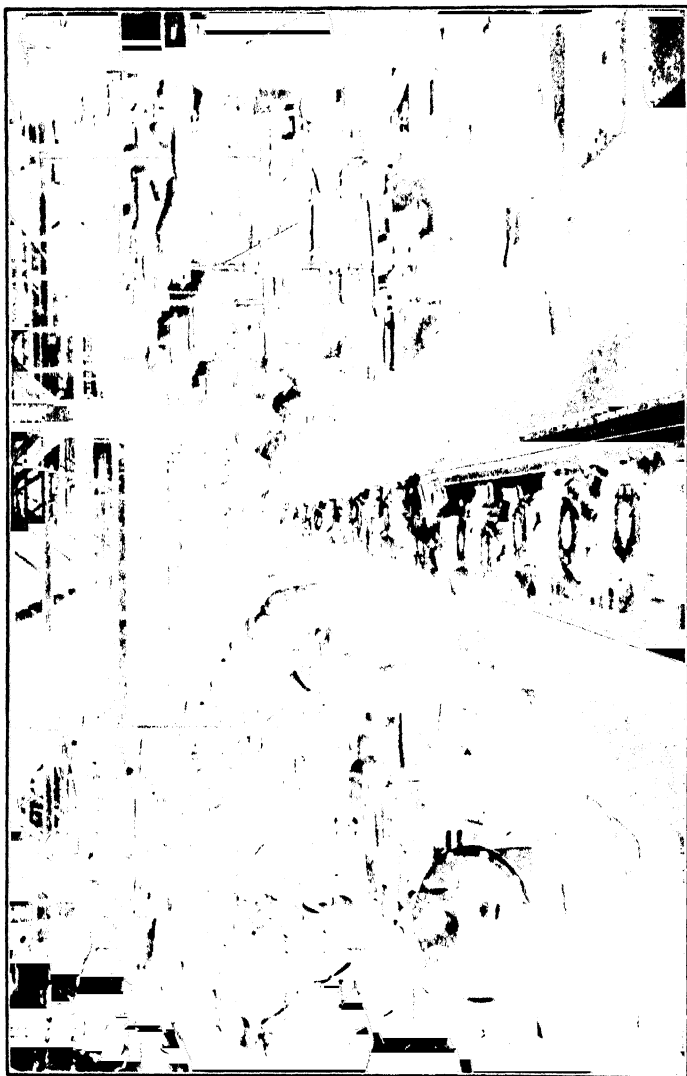
11. If gunpowder consists of 15% charcoal, how much charcoal is required to make up 250 pounds of gunpowder?

12. Plaster is made from a mixture of 5 bushels of lime and 7 bushels of screen sand. What proportion of the mixture is sand?

13. If \$6.00 is 20% of a man's money, how much has he?

14. A train of gears transmits 64% of the power supplied at one end. If 14.3 H. P. is furnished to the train, how much will be delivered to the machinery at the other end? Ans. 9.152 H.P.

15. A certain 200 H. P. steam engine uses only 18.5% of the energy of the coal. What would be its horse-power if it turned all of the energy into useful work? Ans. $1081.08 + \text{H.P.}$



GENERAL VIEW SHOWING MACHINE LAYOUT FOR DRILLING CLUTCH CASE

Courtesy of Ford Motor Company, Detroit, Michigan

PRACTICAL MATHEMATICS

PART III

DENOMINATE NUMBERS

40. A *denominate number* is one in which the unit of value is established by law or custom. For example, 7 pounds, 6 feet, 10 kilograms.

When a denominate number is composed of units of but one denomination, as, for example, 3 gallons, it is called a *simple number*. If it contains units of more than one denomination that are related to each other, as 6 feet 10 inches, or 7 pounds 5 ounces, it is called a *compound number*.

The *reduction* of denominate numbers is the process of changing them from one denomination to another without changing their value. The reduction may be from a higher to a lower denomination, or from a lower to a higher denomination.

NOTE In a decimal system like the *Metric System*, the units increase and decrease by a uniform scale of 10, but in the *English System* the scale varies.

MEASURES

41. A *unit of measure* is a standard by which a quantity—such as length, area, capacity, or weight—is measured. For example, the length of a piece of cloth is ascertained by applying the *yard* or the *meter* measure; the capacity of a cask by the use of the *gallon* or the *liter* measure; the weight of a body by the use of the *pound* or the *kilogram*, etc.

There are two systems of measurement which are legalized in the United States, the *English System* and the *Metric System*. The former is in common use in the United States and England, the latter in all other countries and in our own governmental departments. The metric system is introduced here for general information and for those who hold government positions or who are in foreign trade. The student is referred to the Appendix for a complete list of tables in the English and Metric systems.

MEASURES OF EXTENSION

42. Extension is that property of a body by virtue of which it occupies space and has one or more of the dimensions—length, breadth, and thickness.

A *line* has a single dimension—length—and its measurement is accomplished by *linear measure*.

A *surface* or *area* has two dimensions—length and breadth—and its measurement is accomplished by *square measure*.

A *solid* has three dimensions—length, breadth, and thickness—and its volume or capacity is obtained by *cubic measure*.

The *standard units of extension* in the United States are the *yard* and the *meter*. The yard is 36 inches, the meter 39.37 inches.

43. Linear Measure. The English measure for length or distance, called *long measure*, makes use of the *yard* as its fundamental unit, with subdivisions for convenience into *feet* and *inches*. For instance, a merchant sells cloth by the yard; a person measures his height in feet, and the length of his arm in inches. Larger units, the rod and the mile, are used when distances to be measured become so great that the small units are not convenient. A complete table of long measure is given in the Appendix. For ordinary calculations it is sufficient to remember the following:

$$\begin{aligned} 12 \text{ inches (in.)} &= 1 \text{ foot (ft.)} \\ 3 \text{ ft. or } 36 \text{ in.} &= 1 \text{ yard (yd.)} \\ 5280 \text{ ft.} &= 1 \text{ mile (mi.)} \end{aligned}$$

The *metric system* is founded on the *meter* as the fundamental unit, and as it is a decimal system, the smaller and larger units are all decimal divisions or multiples of the meter. For example, the centimeter, which is about $2\frac{1}{2}$ times smaller than the inch, is, as its name indicates, $\frac{1}{100}$ of a meter; while the kilometer, which is about .6 miles is 1,000 meters. Remember the following units:

$$\begin{aligned} 10 \text{ millimeters (mm.)} &= 1 \text{ centimeter (cm.)} \\ 100 \text{ centimeters (cm.)} &= 1 \text{ meter (m.)} \\ 1000 \text{ meters (m.)} &= 1 \text{ kilometer (km.)} \end{aligned}$$

It is also useful to remember the following approximate values
 25 mm. or 2.5 cm. = 1 inch, Fig. 2; 30 cm. = 1 ft.; 1 kilometer = .6 mi.

Engineers have adopted the *decimal* plan in connection with the English system by using scales and steel tapes with feet divided

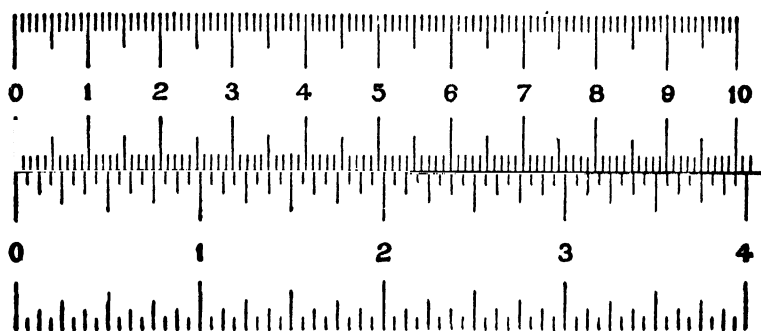


Fig. 2. Comparison between Centimeter and Inch Scales

into tenths. They also use Surveyors' long measure in making land surveys. (See Appendix.)

44. Square Measure. A surface has two dimensions, length and breadth.

The *area* of a surface is defined as the number of units of surface it contains, and is equal to the product of its two dimensions expressed in the same linear units. The *unit of surface* is a square, which is a plane figure bounded by four equal sides and having four right angles, Fig. 3. A square, each side of which is one inch in length, is called a square inch. Squares formed with sides of 1 foot, 1 meter, 1 mile, etc., are called respectively, 1 square foot, 1 square meter, 1 square mile, etc.



Fig. 3.
A Foot
Square.

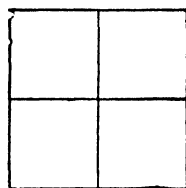


Fig. 4. A 2-Foot
Square.

A distinction should be clearly made between the terms *square foot* and *foot square* or between *square mile* and *mile square*. If Fig. 3 may be supposed to represent a square 1 foot on a side, it may be called either 1 sq. ft. or 1 ft. square. On the other hand, Fig. 4 measures two feet on a side and hence it is 2 ft. square, but, as may readily be seen, it has not 2 sq. ft. but 4 sq. ft. of area. Therefore a cattle ranch covering an area 3 miles square has really 9 square miles of surface.

Square measure, therefore, involves units whose names are the same as those used in linear measure with the term *square* in front of each. The English system has an extra unit used in measurements of land, which is called the *acre*, equal to the area of a square about 209 ft. on a side.

In the metric system, the *square centimeter* and *square meter*—the latter being about 20% larger than the English *square yard*—are used for small areas. The larger surfaces are measured in *ares* and *hectares*, the former being 10 meters square ($\frac{1}{40}$ acre) and the latter equal to 100 ares or $2\frac{1}{2}$ acres.

45. Cubic Measure. The volume of any solid is obtained by *cubic measure*. The unit of volume is a cube, Fig. 5, each edge of which is some unit of length; for example, the cubic inch, cubic centimeter, cubic foot are common units of volume.

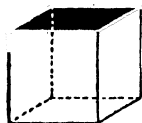


Fig. 5. A 1-Foot Cube.

The volume of a body of rectangular figure is equal to the product of its three dimensions, each expressed in the same linear unit.

In the case of an irregular body, however, the volume although still expressed in cubic measure, must be measured by displacement of water. Liquids are classed as irregular bodies but in the English system are measured by a different unit, giving rise to the classification as given in Sec. 46.

In Sec. 44 it was proved that the area of a square surface increases as the second power of the side of the square; *i. e.*, a surface 2 feet square was found to cover 2×2 or 4 sq. ft. of area. In the same manner, Figs. 5 and 6 show that a 2-foot cube has $2 \times 2 \times 2$ or 8 cubic feet of volume; *i. e.*, the volume of a cube increases as the third power of the length of the side.

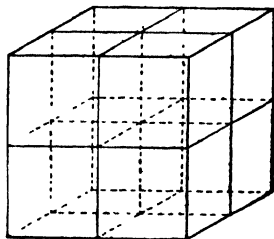


Fig. 6. A 2-Foot Cube.

NOTE. The use of the multiplication sign in finding the volume has given rise to its use in indicating the dimensions of surfaces and solids. Thus a 2×12 joist means a joist 2-inches thick and 12 inches wide; or a room $15' \times 12' \times 10'$ means a room 15 feet long, 12 feet wide, and 10 feet high.

The units of cubic measure in the English system are the same as those in long measure with the prefix *cubic*; for example, cubic inch, cubic yard. The *cord* (128 cu. ft.) for wood and the *perch* (24½ cu. ft.) for stone or masonry are also used.

In the metric system, the *cubic centimeter* and *cubic meter* are common, the latter being 30% larger than the *cubic yard* and used in place of it in measuring earth and rock excavations, as well as in measuring timber, stone, etc., where the English *cord* and *perch* are used.

MEASURES OF CAPACITY

46. *Capacity* signifies the extent of volume or space. In the English system a lack of unity exists in the measurement of capacity because of the use of several kinds of measure. For example, the common *liquid measure* and the *apothecaries' fluid measure* are used for liquids, and still another kind called *dry measure* for grains, vegetables, etc.

This complication is avoided in the metric system by having the same units for all measurements of capacity.

47. **Liquid Measure.** Liquid measure is used in measuring liquids and in estimating the capacity of cisterns, reservoirs, etc. In the English system, the most common liquid measure makes use of the gallon, barrel, etc. The following units should be remembered:

$$\begin{aligned} 2 \text{ pints (pt.)} &= 1 \text{ quart (qt.)} \\ 4 \text{ quarts} &= 1 \text{ gallon (gal.)} \\ 31\frac{1}{2} \text{ gallons} &= 1 \text{ barrel (bbl.)} \end{aligned}$$

In the metric system the unit is the *liter*, which is 5% larger than the liquid quart and 10% smaller than the dry quart, Fig. 7.

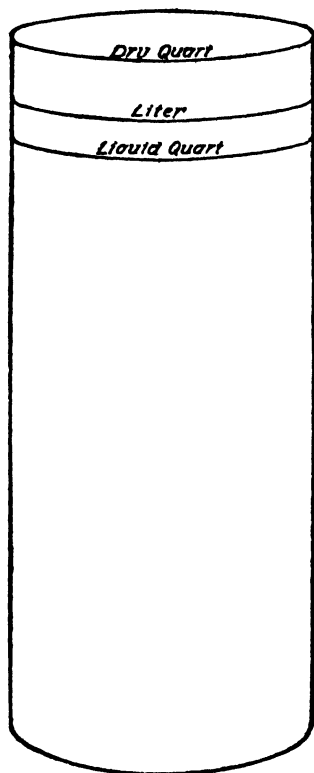


Fig. 7. Comparison of Liter, Dry Quart, and Liquid Quart.
($\frac{1}{2}$ actual size)

Note the following:

$$1000 \text{ cubic centimeters (c.c.)} = 1 \text{ liter (l.)}$$

$$100 \text{ liters} = 1 \text{ hectoliter.}$$

48. Dry Measure. Dry measure is used in measuring dry substances such as grain, vegetables, salt, etc. In the English system the quart, peck, and bushel are used. In the metric system the *hectoliter* serves the same purpose as the United States *bushel* and is equal to about 3 bushels.

The common units in dry measure are:

$$8 \text{ quarts*} = 1 \text{ peck (pk.)}$$

$$4 \text{ pecks} = 1 \text{ bushel (bu.)}$$

MEASURES OF WEIGHT

49. Weight is a measure of the force of the earth's attraction for a body.

In the English system several units of weight are used, viz, the standard *Troy weight*, which is used in weighing gold, silver, and jewels; the more common *avoirdupois weight*, used in general trade; and the *apothecaries' weight*, used by druggists and physicians. The student is referred to the Appendix for the Troy and apothecaries' measure.

50. Avoirdupois Weight. Avoirdupois weight is used to measure the weight of objects in general trade. The most useful units are as follows:

$$16 \text{ ounces (oz.)} = 1 \text{ pound (lb.)}$$

$$2000 \text{ pounds} = 1 \text{ ton (t. or T.)}$$

The ton just given is the *short ton* and is more generally used than the *gross ton* which is equal to 2,240 lbs. This latter unit is now mainly used in the United States custom house and in weighing coarser articles, such as coal at the mine.

51. Metric Weight. In the metric system the unit of weight is the *gram* which is equal to the weight of 1 cubic centimeter of pure water at a temperature of 38° F. The following are the most important units:

$$1000 \text{ grams (g.)} = 1 \text{ kilogram (kg.)}$$

$$1000 \text{ kilograms} = 1 \text{ metric ton.}$$

*The dry quart is about 15% larger than the liquid quart. Fig. 7.

30 grams approximately equal one ounce, and one kilogram equals 2.2 pounds, hence the gram is used wherever the ounce, penny-weight, etc., would be used, while the *kilo* and *half kilo* replace the pound. The *metric ton* serves the same purpose as the *short* or *gross tons*.

MEASURE OF TIME

52. *Time* is measured in the same manner and by the same unit throughout the civilized world, the unit being the *mean solar lay*.

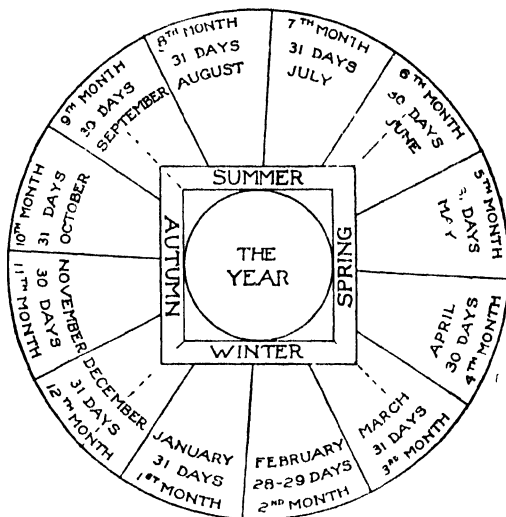


Fig. 8. Chart of the Calendar Year.

This *mean solar day* is obtained by taking the average of all of the days of the year, a day being measured from noon of one day to noon of the next day. This process is necessary because the position of the earth relative to the sun changes as the year advances, and, therefore, no one day can be taken as the true day. Evidently the time when it is *noon*, i. e., the instant the sun is passing the north and south line, will be different as we pass from east to west. For example, London has her noon five hours earlier than New York. This has led to the adoption in the United States of *four standard times*, the *Eastern*, *Central*, *Western*, and *Pacific*, each one hour later than the preceding. Therefore, when it is noon in Washington it is 11 o'clock

in Chicago, 10 o'clock in Denver, and 9 o'clock in San Francisco. All cities or towns falling in one of the four regions adopt the standard time of that section. The divisions of the calendar year are shown graphically in Fig. 8.

MISCELLANEOUS MEASURES

53. The *English money system* has for its principal units the *pound sterling* (£.) equal roughly to \$5.00, and the *shilling*, equal to 25 cents. The *French money system* has for its unit the *franc*, equal roughly to 20 cents. The *German money system* makes use of the *mark*, equal to about 25 cents, for its principal unit.

A useful table of *enumeration* is given briefly as follows:

12 units = 1 dozen (doz.)

12 dozen = 1 gross (gro.)

Another table much used by stationers is as follows:

24 sheets = 1 quire

20 quires = 1 ream

The student is referred to the Appendix for the complete tables to which references have already been made together with others which are in more or less common use. It is well to have these tables fairly well memorized, and problems are given in the following sections on the assumption that the required familiarity has been attained.

REDUCTION OF DENOMINATE NUMBERS

54. The *reduction of denominate numbers* in the English system is accomplished according to the following rules:

To change a compound denominate number to a simple number of lower denomination: *Multiply the integer of the highest denomination by the number of units of the next lower denomination in one unit of the higher denomination, and add to this product the given number of the lower denomination. Proceed into lower terms in this manner until the required denomination is reached.*

To change a simple denominate number to a compound number of higher denominations: *Divide the given number by the number of units contained in one unit of the next higher denomination. Set aside the remainder; then in the same manner divide the quotient thus*

obtained, and proceed in this way until the required denomination is reached. The last quotient and the several remainders will be the result sought.

Examples. 1. Reduce 6 mi. 116 yds. 24 ft. to feet.

SOLUTION. Since 1 mile = 1760 yds., 6 mi. = 6×1760 yds. = 10,560 yds.; and with 116 yds. added this becomes 10,676 yds. Since 1 yd. = 3 ft., 10,676 yds. = 32,028 ft., and with 24 ft. added, this becomes 32,052 feet, Ans.

2. Reduce 765 liquid pints to higher denominations.

SOLUTION. Dividing 765 pints by 2 gives 382 qts. 1 pt. Dividing by 4 to reduce to gallons gives 95 gal. 2 qts. 1 pt. Dividing by 31.5 to reduce to bbl. gives 3 bbl. $\frac{1}{2}$ gal. 2 qts. 1 pt. As 2 qts. = $\frac{1}{2}$ gal. this added to the $\frac{1}{2}$ gal. already obtained gives as a final result 3 bbl. 1 gal. 1 pt.

$$\begin{array}{r} 2)765 \text{ pts.} \\ 4)382 \text{ qts.} + 1 \text{ pt.} \\ 31.5)95 \text{ gal.} + 2 \text{ qts.} \\ \hline 3 \text{ bbl.} + \frac{1}{2} \text{ gal.} \end{array}$$

55. Reduction in the metric system is accomplished *by moving the decimal point.*

Example. Reduce 10,450 millimeters, 276 centimeters, and 600 meters to kilometers.

SOLUTION. Reduce 10,450 mm. to meters by moving the decimal point three places to the left = 10.45 meters. Similarly reduce 276 cm to m. by moving the decimal point two places to the left = 2.76 m. $600 + 2.76 + 10.45 = 613.21$ m. This result may be changed to kilometers by moving three points to the left = 0.6132 km.

When quantities expressed in the English system are to be reduced to their equivalents in the metric system, or the reverse, the comparative tables, p. 105, are used. For example, if 34 meters should be changed to yards, the table shows that 1 meter equals 1.1 yards; 34 meters equal 34×1.1 yards or 37.4 yards.

Examples. 1. Change 172 feet to centimeters.

SOLUTION. From the table, 1 foot equals .3 m. 172 feet equals $172 \times .3$ or 51.6 m. 51.6 m. equals $51.6 \times 100 = 5160$ cm

2. Change 12.4 pounds to grams.

SOLUTION. From the table, 1 pound equals .45 kilograms. 12.4 pounds equals $12.4 \times .45$ or 5.58 kilograms. 5.58 kilograms equals 5.58×1000 or 5580 grams.

PROBLEMS FOR PRACTICE

1. Change 2560 lbs to ounces; to tons.
2. Change 42 meters to centimeters; to millimeters.

3. Reduce 76 yds. to inches; to feet.
4. Change 35 ft. to centimeters (approx.).

Reduce

5. 18 lbs. 12 oz. to ounces.
6. £ 8 12 s to shillings.
7. 75 m. 86 cm. to centimeters.
8. 8 bu. 4 pks. to pecks.
9. 3 A. 78 sq. ft. to square feet.
10. 5 bbl. 12 gal. to gallons.
11. 2½ T. to pounds.
12. \$746.50 to dimes.

Reduce to higher denominations

13. 1560 ft.
14. 15,760 mm.
15. 86,400 sec.
16. 1640 dry pints.
17. 685 doz.
18. 659 shillings.
19. 7562 oz., avoird.
20. 375 pints.
21. 2500 sq. in.

Using approximate values, change

22. 120 yds. to meters.
23. 15 s. to dollars.
24. 56 kilos to pounds.
25. 4750 francs to dollars.
26. 65 cm. to inches.
27. 450 liters to pints.

OPERATIONS WITH DENOMINATE NUMBERS

56. There are two methods of *adding, subtracting, multiplying, or dividing* denominate numbers. One is to reduce the given numbers to the lowest denomination mentioned in the example, then perform the required process and reduce the result to higher denominations. The other is to perform the process on the numbers as they stand, making the necessary reductions during the operations.

57. Addition and Subtraction. Examples. 1. Find the sum of 3 mi. 182 rd. 4 yds. 2 ft.; 304 rd. 1 ft.; and 5 mi. 76 rd. 4 yds. 2 ft.

SOLUTION a.

$$\begin{aligned}
 3 \text{ mi. } 182 \text{ rd. } 4 \text{ yds. } 2 \text{ ft.} &= 15,840 + 3,003 + 12 + 2 = 18,857 \text{ ft} \\
 304 \text{ rd. } 0 \text{ yds. } 1 \text{ ft.} &= 5,016 + 1 = 5,017 \text{ ft.} \\
 5 \text{ mi. } 76 \text{ rd. } 4 \text{ yds. } 2 \text{ ft.} &= 26,400 + 1,254 + 12 + 2 = 27,668 \text{ ft.} \\
 &51,542 \text{ ft.}
 \end{aligned}$$

$$51,542 = 9 \text{ mi. } 243 \text{ rd. } 4 \text{ yds. } 0 \text{ ft. } 6 \text{ in. Ans.}$$

SOLUTION. b. The sum of the right-hand column = 5 ft. = 1 yd. 2 ft.; write down the 2 ft. and add 1 to the yds. column. The sum of the yds. column and 1 carried = 9 yds.; $9 + 5\frac{1}{2} = 1 \text{ rd. } + 3\frac{1}{2} \text{ yds.}$, the 1 rd. to be added to the rds. column. The sum of the rds. column + 1 carried = 563 rds. $563 \div 320 = 243 \text{ rds.} + 1 \text{ to be added}$

mi.	rd.	yds.	ft.
3	182	4	2
0	304	0	1
5	76	4	2
9	243	4	½

to the mi. column, making it 9 miles. The result is therefore 9 mi. 243 rds. 3½ yds. 2 ft. = 9 mi. 243 rds. 4 yds. 0 ft. 6 in.

2. Subtract 3 pks. 1 qt. 1 pt. from 1 bu. 2 pks. 4 qts.

SOLUTION. Mentally take 1 qt. (2 pts.) from the second column and place it in the first; $2 - 1 = 1$ pt. $1 \text{ bu. } (4 \text{ pks.}) + 2 \text{ pks.} = 6$; $6 - 3 = 3$ pks. The result is therefore 3 pks. 2 qts. 1 pt.

bu.	pks.	qts.	pts.
1	2	4	0
0	3	1	1
	3	2	1

PROBLEMS FOR PRACTICE

Find the sum of

1. 10 yds. 2 ft. 10 in.; 15 yds. 1 ft. 9 in.; 8 yds. 2 ft. 7 in.; 18 yds. 1 ft. 11 in.; 16 yds. 2 ft. 8 in. Ans. 12 rd. 4 yds. 2 ft. 9 in.

2. 12 A. 35 sq. rd.; 14 A. 110 sq. rd.; 15 A. 132 sq. rd.; 11 A. 96 sq. rd.; 25 A. 100 sq. rd. Ans. 79 A. 153 sq. rd.

3. 5 t. 6 cwt. 14 lbs. 10 oz.; 7 t. 15 cwt. 36 lbs. 15 oz.; 17 t. 5 cwt. 84 lbs. 12 oz.; 70 t. 9 cwt. 94 lbs. 11 oz.

Ans. 100 t. 17 cwt. 31 lbs.

From

4. 12 gal. 2 qts. 1 pt. 2 gi. take 6 gal. 3 qts. 1 pt. 3 gi.

Ans. 5 gal. 2 qts. 1 pt. 3 gi.

5. 15 yds. 2 ft. 7 in. take 4 yds. 2 ft. 10 in.

Ans. 10 yds. 2 ft. 9 in.

6. 25 t. 8 cwt. 75 lbs. 10 oz. take 10 t. 11 cwt. 35 lbs. 15 oz.

Ans. 14 t. 17 cwt. 39 lbs. 11 oz.

58. Multiplication and Division. Examples. 1. Multiply 14 gals. 3 qts. 1 pt. by 7.

Solution a.

14 gal. 3 qts. 1 pt. = $112 + 6 + 1 = 119$ pts.

$119 \text{ pts.} \times 7 = 833 \text{ pts.} = 416 \text{ qts. } 1 \text{ pt.} = 104 \text{ gal. } 1 \text{ pt.}$ Ans.

SOLUTION b. $(14 \text{ gal. } 3 \text{ qt. } 1 \text{ pt.}) \times 7 = 98 \text{ gal.}$ 14 gal. 3 qts. 1 pt.
 $21 \text{ qts. } 7 \text{ pts.} = 98 \text{ gal. } 24 \text{ qts. } 1 \text{ pt.} = 104 \text{ gal. } 0 \text{ qts.}$ 7
 1 pt. Ans. 104 gal. 0 qts. 1 pt.

2. Divide 14 gal. 3 qts. 1 pt. by 4.

Solution a.

14 gal. 3 qts. 1 pt. = 119 pts.

$4 \overline{)119} \text{ pts.}$

$29 \frac{3}{4} \text{ pts.} = 14 \text{ qts. } 1 \frac{3}{4} \text{ pts.} = 3 \text{ gal. } 2 \text{ qts. } 1 \frac{3}{4} \text{ pts.}$ Ans.

PROBLEMS FOR PRACTICE

Multiply

1. 3 hrs. 20 min. 35 sec. by 5. Ans. 16 hrs. 42 min. 55 sec.
2. 2 t. 5 cwt. 48 lbs. 15 oz. by 8. Ans. 18 t. 3 cwt. 91 lbs. 8 oz.
3. 12 cu. yds. 15 cu. ft. by 6. Ans. 75 cu. yds. 9 cu. ft.

Divide

4. 15 bu. 3 pks. 5 qts. by 4. Ans. 3 bu. 3 pks. 7 qts. $\frac{1}{2}$ pt.
5. 23 cwt. 68 lbs. 10 oz. by 5. Ans. 4 cwt. 73 lbs. $11\frac{3}{8}$ oz.
6. 15 rd. 4 yds. 2 ft. 8 in. by 5. Ans. 3 rd. 2 ft. $11\frac{1}{4}$ in.

POWERS AND ROOTS

59. Powers. A *power* of a quantity is the product of factors, each of which is equal to that number. This quantity may be simply a number, as 1, 2, 3, or 4, or it may be any letter, as *a*, *b*, *c*, or *d*, which may have any numerical value whatsoever.

In order to show how many times the quantity is to be used as a factor, a small figure is placed to the right and a little above the quantity; as 3^4 or a^4 . This small number is called an *exponent*, and shows to what power the quantity is to be raised. Thus, 3^4 means the fourth power of 3, or $3 \times 3 \times 3 \times 3 = 81$; *i. e.*, 3 is taken 4 times as a factor. If the letter *a* is substituted for the figure 3 and this letter is raised to the fourth power, the result is $a \times a \times a \times a$ or a^4 . Since no definite value for *a* has been given, the result of raising the letter to the power can only be indicated; thus, a^4 .

The *second* power of the quantity is called its *square*. For example, 4 is the square of 2, for $2 \times 2 = 4$; again, a^2 is the square of *a*, for $a \times a = a^2$. The third power of a quantity is called its *cube*; thus, 8 is the cube of 2, for $2 \times 2 \times 2 = 8$; or a^3 is the cube of *a*.

Suppose now it is required to find the square of a^2 . $a^2 \times a^2 = a^4$, for $a^2 \times a^2 = (a \times a) \times (a \times a) = a^4$; or, in other words, *when two like quantities are multiplied together, the exponent of the product is equal to the sum of the exponents of the quantities.*

The power of a fraction is obtained by multiplying the numerator by itself and the denominator by itself the required number of times.

Thus, the second power of $\frac{2}{4} = \frac{2 \times 2}{4 \times 4} = \frac{4}{16}$; the third power of

$$\frac{1}{7} = \frac{1 \times 1 \times 1}{7 \times 7 \times 7} = \frac{1}{343}.$$

Examples. 1. What is the cube, or third power of 21?

$$21^3 = 21 \times 21 \times 21$$

$$\begin{array}{r} 21 \\ 21 \\ \hline 21 \end{array}$$

SOLUTION. Here it is readily seen that raising 21 to the third power is the same as using 21 three times as a factor.

$$\begin{array}{r} 42 \\ 441 \\ 21 \\ \hline 441 \\ 882 \\ \hline 9261 \end{array}$$

Ans.

2. What is the cube, or third power of $2a$?

$$(2a)^3 = 2a \times 2a \times 2a.$$

SOLUTION. Note here, that when raising a quantity like $2a$ to a power, the coefficient 2 and the letter a are each raised separately to the required power.

$$\begin{array}{r} 2a \\ 2a \\ \hline 4a^2 \\ 2a \\ \hline 8a^3 \end{array}$$

Ans.

3. What is the cube of .71?

$$.71^3 = .71 \times .71 \times .71$$

$$\begin{array}{r} .71 \\ .71 \\ \hline .71 \end{array}$$

SOLUTION. When raising a decimal like .71 to a power, care should be taken to correctly place the decimal point. In the answer to this problem there will be six places to be pointed off, thus bringing the decimal point before the 3.

$$\begin{array}{r} 497 \\ .5011 \\ .71 \\ \hline .5041 \\ 35287 \\ \hline .357911 \end{array}$$

Ans.

4. What is the fourth power of $\frac{3}{5}$?

$$\frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{3 \times 3 \times 3 \times 3}{5 \times 5 \times 5 \times 5} = \frac{81}{625}.$$

Ans.

PROBLEMS FOR PRACTICE

Raise the following quantities to the power indicated:

1. 47 to the second

Ans. 2,209

2. $6m$ " " second

- | | | | | |
|----|-------|----|------------|----------------|
| 3. | 71 | to | the second | Ans. 5,041 |
| 4. | 2.61 | " | " second | Ans. 6.8121 |
| 5. | .13 | " | " third | |
| 6. | c^3 | " | " third | |
| 7. | r | " | " fifth | |
| 8. | k | " | " fourth | |
| 9. | $6k$ | " | " fourth | Ans. $1296k^4$ |

60. Roots. A *root* of a quantity is one of the equal factors which, when multiplied together, give the quantity. Thus, if a certain quantity is used twice in order to produce another quantity, then the quantity first mentioned is the *square root* of the second. Thus, 2 is the square root of 4, for $2 \times 2 = 4$. If the first quantity must be used three times as a factor, it is the *cube root* of the second quantity; thus, the cube root of 8 is 2, for $2 \times 2 \times 2 = 8$; the cube root of 27 is 3, for $3 \times 3 \times 3 = 27$. This also applies to any root as the 4th, 5th, etc. Thus, the 5th root of $a^5 = a$, for $a \times a \times a \times a \times a = a^5$.

This process of finding the root is merely the reverse of finding the power, and is termed *extracting the root*.

The *radical sign* $\sqrt{\quad}$ when placed before a quantity shows that some root of it is to be taken. The root is indicated by a small figure called the *index* placed above the radical sign; thus, $\sqrt[8]{\quad}$. When no index is written, the square root is always understood.

The following examples show the meaning of the sign and index:

$$\sqrt{81} = 9, \text{ for } 9 \times 9 = 81.$$

$$\sqrt[3]{27} = 3, \text{ for } 3 \times 3 \times 3 = 27.$$

$$\sqrt[4]{81} = 3, \text{ for } 3 \times 3 \times 3 \times 3 = 81.$$

$$\sqrt[5]{32} = 2, \text{ for } 2 \times 2 \times 2 \times 2 \times 2 = 32.$$

$$\sqrt{b^2} = b, \text{ for } b \times b = b^2.$$

$$\sqrt[3]{8b^3} = 2b, \text{ for } 2b \times 2b \times 2b = 8b^3.$$

$$\sqrt[5]{32b^{10}} = 2b^2, \text{ for } 2b^2 \times 2b^2 \times 2b^2 \times 2b^2 \times 2b^2 = 32b^{10}.$$

61. Square Root. In the above illustrations the required root may be readily determined by inspection, but in most cases, this method cannot be followed. When the root desired is the *square root* the method illustrated below should be used.

Examples. 1. Find the square root of 185,761.

SOLUTION. First point off the number into periods of two figures each commencing at the right of the number. Then find the largest number whose square is equal to or less than 18. This is found to be 4. Set down the square of 4 or 16 under 18, and place 4 as the first figure of the root.

Subtract 16 from 18 and to the remainder annex the next period, obtaining 257 as the new dividend. The new divisor is twice the root already found with a cipher annexed, that is $2 \times 40 = 80$. By trial 80 is found to be contained 3 times in 257, so 3 is placed as the second figure of the root. Add 3 to 80 and multiply this sum 83 by 3. Place this product, 249, under 257 and subtract. Annex to the remainder the last period obtaining 861 as the new dividend.

The next divisor is twice 430 or 860. This is contained in 861 once. Place the 1 as the third figure of the root, add 1 to 860, thus making the complete divisor 861. **As there is no remainder, the number 185761 is a perfect power, and its square root is 431.**

$$\begin{array}{r}
 431 \\
 2 \times 40 = 80 \quad 18 \quad 57 \quad 61 \\
 \underline{16} \\
 80)257 \\
 \underline{249} \\
 80)861 \\
 \underline{861}
 \end{array}$$

2. Find the square root of 18,763.8910.

SOLUTION. Point off the periods to the left and right of the decimal point, obtaining for the whole number the periods, 1, 87, and 63, and for the decimal the periods, 89 and 10. The root must have three numerical places to the left of the decimal point to correspond to the three periods in the whole number; hence, the final value of the root is 136.98.

$$\begin{array}{r}
 136.98+ \\
 2 \times 10 = 20 \quad 1 \quad 87 \quad 63.89 \quad 10 \\
 \underline{3 \quad 1} \\
 20)87 \\
 \underline{60} \\
 20)27 \\
 \underline{20} \\
 20)63.89 \\
 \underline{40} \\
 20)23.89 \\
 \underline{20} \\
 20)3.89 \\
 \underline{20} \\
 20)18.763.8910 \\
 \underline{160} \\
 20)2763.8910 \\
 \underline{2720} \\
 20)438910 \\
 \underline{4360} \\
 20)2910 \\
 \underline{2900} \\
 20)10 \\
 \underline{10}
 \end{array}$$

3. Find the square root of .001225.

SOLUTION. In finding the square root of a decimal like the above, it is seen that the first period contains only zeros; consequently the first figure of the root will be a zero. Begin by finding the square next smaller than 12, and proceed as in previous examples. Point off three places in the root to correspond to the three periods in the decimal.

$$\begin{array}{r} .035 \\ .00 \ 12 \ 25 \\ \underline{9} \\ 65 \overline{)325} \\ \underline{325} \end{array}$$

4. Find the square root of 3 to three decimal places.

SOLUTION. Annex ciphers at the right of the decimal point and separate into periods as usual.

$$\begin{array}{r} 1.732 + \\ \overline{3.00 \ 00 \ 00} \\ 1 \\ 27 \overline{)200} \\ \underline{189} \\ 343 \overline{)1100} \\ \underline{1029} \\ 3462 \overline{)7100} \end{array}$$

Rules. (a) *Point off the given number into periods of two figures, beginning at the decimal point. If the number contains a whole number and a decimal, point off periods to the left of the decimal point for the whole number, and to the right for the decimal, annexing a cipher to the last figure of the decimal if necessary.*

(b) *Find the greatest number whose square is contained in the left-hand period. This figure will be the first figure in the root. Subtract the square of this number from the left-hand period and annex the next period to the remainder to form the new dividend.*

(c) *After annexing a cipher to that part of the root already found and doubling the result, place the quantity thus obtained on the left for a trial divisor. Ascertain how many times this divisor is contained in the new dividend, and write the quotient as the next figure of the root. Then add the number just placed in the root to the trial divisor, and multiply the completed divisor by the last figure in the root. Subtract this product from the dividend, and to the remainder annex the next period for the new dividend. Continue the operation until all the periods are used.*

(d) *To place the decimal point in the result, if the number is a decimal number, point off in the root as many figures from the left as there are periods from the left to the decimal point in the original number.*

(e) *If after the root is found there is a remainder, it should be dropped and a + sign placed after the root.*

PROBLEMS FOR PRACTICE

- | | |
|---------------------------------------|--------------|
| 1. Find the square root of 364,936 | Ans. 604.09+ |
| 2. Find the square root of 825,487.69 | Ans. 908.56+ |
| 3. Find the square root of .003864 | Ans. .0621+ |
| 4. Find the square root of .0563 | Ans. .2372+ |
| 5. Find the square root of 1,873 | |
| 6. Find the square root of 6,432 | |

62. Square Root of Fractions. When the square root of a common fraction is desired, first see if the numerator and denominator have perfect square roots. If so, write the root of the numerator over the root of the denominator. If this is not the case, reduce the common fraction to decimal form and find the required root as before. If the quantity is a mixed quantity change to decimal form before extracting the root.

The following solutions will illustrate these principles:

Examples. 1. Find the square root of $\frac{16}{49}$

$$\sqrt{\frac{16}{49}} = \frac{\sqrt{16}}{\sqrt{49}} = \frac{4}{7} \quad \text{Ans.}$$

2. Find the square root of $\frac{4a^2}{16b^2}$

$$\frac{\sqrt{4a^2}}{\sqrt{16b^2}} = \frac{2a}{4b} \quad \text{Ans.}$$

3. Find the square root of $\frac{169}{250}$ to three decimal places.

$$\begin{array}{r} 250 \overline{)169.000} \\ \underline{.676} \end{array} \qquad \begin{array}{r} .822 + \\ .67 \overline{)60.00} \\ \underline{64} \\ 162 \overline{)360} \\ \underline{324} \\ 1642 \overline{)3600} \\ \underline{3284} \end{array}$$

$$\text{Hence, } \sqrt{\frac{169}{250}} = \sqrt{.676} = .822 + \quad \text{Ans.}$$

4. Find the square root of $\frac{64k^2}{81m^2}$

$$\sqrt{\frac{64k^2}{81m^2}} = \frac{8k}{9m} \quad \text{Ans.}$$

5. Find the square root of $\frac{3}{8}$ to two decimal places.

$$\begin{array}{r} 8 \overline{)3.000} \\ \underline{.375} \end{array} \qquad \begin{array}{r} .61 + \\ .37 \overline{)50} \\ \underline{36} \\ 121 \overline{)150} \\ \underline{121} \\ 29 \end{array}$$

$$\text{Hence, } \sqrt{\frac{3}{8}} = \sqrt{.375} = .61 + \quad \text{Ans.}$$

PROBLEMS FOR PRACTICE

Find the square root of

1. $\frac{49}{169}$

4. $\frac{5}{8}$ Ans. $.79 +$

2. $\frac{27}{15625}$ Ans. $.041 +$

5. $\frac{4}{9}$

3. $\frac{64}{81}$

6. $\frac{81a^6}{121b^4}$

RATIO AND PROPORTION

63. Ratio. Ratio is the relation which one quantity bears to another quantity of the same kind; or, in other words, it is the quotient obtained by dividing the one by the other. The ratio of 15 to 3 is 5, because 15 contains 3, five times. The ratio of 3 to 15 is $\frac{1}{5}$, because 3 is $\frac{1}{5}$ of 15.

The two quantities given are called the *terms* of the ratio. The first term is called the *antecedent*; the succeeding one, the *consequent*. The antecedent is the dividend, the consequent is the divisor, and the value of the ratio is the quotient resulting from the division. The symbol used to express a ratio is two dots (:), which is merely an

abbreviated form of the sign of division ($\div$) For example, the ratios 7 to 11, and a to b , are written thus, 7 : 11, and $a : b$. The same ratios may also be expressed in fractional form thus, $\frac{7}{11}$ or $\frac{a}{b}$.

A ratio like the above is called a *direct ratio*; that is, one in which the antecedent is divided by the consequent. An *inverse ratio* is a direct ratio inverted. Thus, if $\frac{7}{11}$ and $\frac{a}{b}$ are the direct ratios, the inverse ratios of the same quantities are $\frac{11}{7}$ and $\frac{b}{a}$.

As a ratio is always an abstract number, it is possible to have a ratio *between similar quantities only*. For example, it is possible to establish a ratio between 75 rivets and 16 rivets, but no ratio between 75 rivets and 10 horses. If the ratio $\frac{a}{b}$ is given, and a represents 5 horses and b , 10 horses, it can be seen that, although this ratio has been expressed in letters, each letter is a quantity of the same kind.

The value of a ratio is not changed by either dividing or multiplying both terms of the ratio by the same number. Thus, for example, if 16 : 32 is divided by 2, the ratio becomes 8 : 16; likewise, each term may be multiplied by 2, obtaining 32 : 64. From the above it is seen that the values of the ratios have not been changed for $\frac{16}{32} = \frac{8}{16} \times \frac{2}{2} = \frac{32}{64} \div \frac{2}{2}$. Consider the ratio $5a : 10b$. Dividing each term by 5, the ratio $a : 2b$ is obtained. Again multiplying each term of the first ratio by 2, it becomes $10a : 20b$, which has the same value as before because $\frac{a}{2b} \times \frac{10}{10} = \frac{10a}{20b}$.

PROBLEMS FOR PRACTICE

What is the value of each of the following?

1. 51 : 17

Ans. 3

2. $20c : 40d$

Ans. $\frac{c}{2d}$

3. 16 : 48

4. $6k : 8m$

Ans. $\frac{3k}{4m}$

5. What is the inverse ratio of $9 : 81$?

Ans. $\frac{81}{9} = 9$

6. What is the inverse ratio of $a : b$?

7. What is the inverse ratio of $10k : 20r$?

64. **Proportion.** (a) A *proportion* is an expression of equality between ratios and is indicated by the sign $::$ or by the equality sign $=$. For example, the proportion $3 : 9 :: 6 : 18$ is read 3 is to 9 as 6 is to 18. The same proportion may be stated thus: 3 over 9 equals 6 over 18; that is, $\frac{3}{9} = \frac{6}{18}$. Any proportion whatever existing between letters is expressed in a similar manner; thus, $a : b :: c : d$ may also be written $\frac{a}{b} = \frac{c}{d}$.

(b) The *terms* of a proportion are made up of the antecedents and consequents of two ratios. Thus, in the proportion $\frac{a}{b} = \frac{c}{d}$, a, b, c, d , are called the *terms* and are numbered—first, second, third, and fourth. This may be illustrated as follows:

First		Second		Third		Fourth
3	:	9	::	6	:	18
or a	:	b	::	c	:	d

The first and fourth terms are called the *extremes*; the second and third terms, the *means*; thus, 3, 18, and a, d , are the extremes, while 9, 6, and b, c are the means.

(c) In all proportions the *product of the means is equal to the product of the extremes*. Thus, in the proportion $3 : 9 = 6 : 18$, the product of the means is $9 \times 6 = 54$ and the product of the extremes is $3 \times 18 = 54$. If the proportion is expressed in fractional form, that is, $\frac{3}{9} = \frac{6}{18}$, it is seen that the law just stated means that the product of

the numerator of the first fraction with the denominator of the second is equal to the product of the numerator of the second fraction with

the denominator of the first. Therefore, in the proportion $\frac{3}{9} = \frac{6}{18}$,

$3 \times 18 = 9 \times 6$ and, in the proportion $\frac{a}{b} = \frac{c}{d}$, $ad = bc$. When

taken by themselves the letters, a, b, c, d , may have any values

whatsoever, but as soon as they are placed in the proportion, their relation to each other becomes fixed. If the proportion has an unknown term to be found, this latter form $\frac{a}{b} = \frac{c}{d}$ is perhaps the simpler form in which to arrange the ratios for finding the missing term.

Rules. (d) *The product of the means divided by either extreme gives the other extreme; or the product of the numerator of the second fraction with the denominator of the first fraction divided by the numerator of the first fraction gives the denominator of the second fraction.*

(e) *The product of the extremes divided by either mean gives the other mean; or the product of the numerator of the first fraction with the denominator of the second fraction divided by the denominator of the first fraction gives the numerator of the second fraction.*

In stating a proportion in which one term is unknown, let x represent the unknown term and solve for x .

Examples. 1. Solve for x in the proportion, $18 : 6 = 9 : x$,
or $\frac{18}{6} = \frac{9}{x}$.

SOLUTION. Product of extremes, $18 \times x = 9 \times 6$, product of means

$$\text{Dividing each side by } 18 \quad x = \frac{54}{18} = 3 \quad \text{Ans.}$$

Hence the missing term is 3.

2. Solve the proportion, $6 : 18 = x : 3$.

SOLUTION. $18 \times x = 6 \times 3 \quad x = 1 \quad \text{Ans.}$

3. Solve the proportion, $9 : 3 = 18 : x$

SOLUTION. $x = \frac{18 \times 3}{9} = 6 \quad \text{Ans.}$

4. Solve the proportion, $3 : 9 = 6 : x$

SOLUTION. $x = \frac{6 \times 9}{3} = 18 \quad \text{Ans.}$

5. Solve for x in the proportion $a : b :: c : x$

SOLUTION. This may be written in this form, $ax = bc$, from which $x = \frac{bc}{a}$. In this case the answer can only be indicated, but if a , b , and c , had the respective values, 5, 6, and 10, their substitution would give to x a value of 12, since $x = \frac{6 \times 10}{5} = 12$.

$$\begin{aligned} ax &= bc \\ x &= \frac{bc}{a} \end{aligned}$$

Find the value of x in the following proportions:

6. $k : m :: x : n.$

SOLUTION. $\frac{k}{m} = \frac{x}{n}$ $mx = kn$ $x = \frac{kn}{m}$ Ans.

7. $s : r :: x : p.$

SOLUTION. $\frac{s}{r} = \frac{x}{p}$ $rx = sp$ $x = \frac{sp}{r}$ Ans.

8. $e : f :: x : h.$

SOLUTION. $\frac{e}{f} = \frac{x}{h}$ $fx = eh$ $x = \frac{eh}{f}$ Ans.

PROBLEMS FOR PRACTICE

Solve for the unknown value of x in the following proportions:

1. $6 : 3 :: m : x$ Ans. $\frac{m}{2}$

6. $21m : 3 :: x : 7$

2. $8 : 9 :: x : 9$

7. $x : 4 :: 3r : s$ Ans. $\frac{12r}{s}$

3. $9 : 81 :: 2 : x$

8. $384 : 64 :: x : m$

4. $4 : x :: 10 : 5$

9. $480 : x :: 10 : 48$

5. $r : s :: 2 : x$

10. $90 : x :: 9 : k$ Ans. $10k$

65. Any root of both sides of a proportion may be extracted, or any power of both sides may be taken, without destroying the proportion.

Examples. 1. Extract the square root of both sides of the following proportion: $1 : 9 = 49 : 441$.

SOLUTION. Extracting the square root of each term

$$1 : 3 = 7 : 21$$

or

$$\frac{1}{3} = \frac{7}{21}$$

which equation, by clearing of fractions, shows that the equality of the proportion has not been destroyed.

2. Extract the square root of both sides of the following proportion: $a^2 : b^2 = c^2 : d^2$.

SOLUTION. This proportion may be stated thus:

$$\frac{a^2}{b^2} = \frac{c^2}{d^2}$$

Extracting the square root $\frac{a}{b} = \frac{c}{d}$

3. Find the cube of the proportion, $1 : 2 = 2 : 4$.

SOLUTION. Changing its form $\frac{1}{2} = \frac{2}{4}$

Cubing each term $\frac{1}{8} = \frac{8}{64}$

Thus it is seen that the proportion is not destroyed.

4. Cube the proportion, $\frac{a}{b} = \frac{c}{d}$

SOLUTION. Cubing each term $\frac{a^3}{b^3} = \frac{c^3}{d^3}$

Thus it is seen the rules are true for both letters and numbers.

66. Two numbers or quantities are *directly proportional* when they increase or decrease together; in which case their *ratio* is always the same.

Two numbers are *inversely proportional* when one increases as the other decreases; in which case their *product* is always the same.

For example, if 8 men perform 10 units of work per hour, they will perform 40 units in 4 hours and 100 units in 10 hours; that is, as the time increases the amount of work done increases. This is a *direct ratio*. Now, on the other hand, if there are say 800 units of work to be done on a job, 8 men will finish this work in 10 hours, 16 men will finish it in 5 hours, 40 men in 2 hours. Here the time varies inversely, or oppositely, as the number of men employed, and such a ratio is called an *inverse ratio*.

67. The following rule will solve any problem in simple proportion whether it be direct or inverse:

Rule. Place for the third term of the proportion the number whose units are like the answer sought (80 rivets in Example 1 below; 7 hours in Example 2). From the conditions of the problem determine if the answer (x) is to be greater or less than the third term; if greater, place the larger of the two remaining numbers for the second term; if less, place it for the first term. Then solve according to the rules given for finding the missing term.

Examples. 1. If a joint 16 feet long requires 80 rivets, what number will a joint 11 feet long require?

SOLUTION. Let x equal the number of rivets required.

Then $16 : 11 = 80 : x$

$$\begin{aligned} \text{or} \quad & \frac{16}{11} = \frac{80}{x} \\ & x = \frac{80 \times 11}{16} \\ & x = 55 \text{ rivets. Ans.} \end{aligned}$$

The result shows that the shorter joint requires fewer rivets; hence, this is called a *direct proportion*.

2. A train running 27 miles per hour covers a certain distance in 7 hours. How long does it take a train running at 32 miles per hour to cover the same distance?

SOLUTION. It is evident that the train running 32 miles per hour will require less time to cover the same distance than the train running 27 miles per hour. Thus, the faster train requires less time, and the slower train more time; or, in other words, the speed varies inversely as the time.

$$32 : 27 :: 7 : x$$

$$32x = 27 \times 7$$

$$x = \frac{189}{32}$$

$$x = 5.9 + \text{hrs. Ans.}$$

It can readily be seen that the proportion $32 : 27 :: 7 : x$ is an inverse one. If the speed varied as the time, the proportion would be direct and stated as follows: $32 : 27 :: x : 7$. It will be noted in the direct form of this proportion just given, that the second and fourth terms refer to the first train and its time, while the first and third terms refer to the other train and the unknown time. In the inverse proportion the time of the first train is found as the third term.

PROBLEMS FOR PRACTICE

1. If the earth moves through 360 degrees in $365\frac{1}{4}$ days, how far will it move in a lunar month of $29\frac{1}{2}$ days? Ans. $29.07 +$

2. How long will it take a gang of 50 workmen to erect the walls of an 8 story building, if it requires 4 days for 100 workmen to erect one story?

3. If 15 men can build a wall 12 feet high in one week, how many men will it require to raise it 20 feet in the same time?

4. If it requires 18 hours to saw 10,000 feet of lumber, using a 20 h.p. engine, what horsepower will be required to saw the same amount in 10 hours?

68. Negative Quantities. If an ordinary thermometer is consulted it will be found that the scale divisions have opposite them numbers which increase both upwards and downwards from 0.

All values below the zero point are considered negative and all values above are considered positive; thus a temperature of -6 degrees means that the mercury reads 6 degrees below zero; similarly 6 degrees means that the reading is 6 degrees above zero.

The boiling point of water on the Fahrenheit thermometer scale is 212 degrees. On the other hand, the temperature of liquid air, which is a very cold body, is -292 degrees Fahrenheit. Thus it is seen that the 0 point is merely a point of reference. In the same manner, the 0 of the numerical system is considered as a figure of no value and all numbers with the $+$ sign have positive values and all numbers with the $-$ sign have negative values.

Suppose a man's money in the bank is 300 dollars and his debts amount to 350 dollars; here his money in the bank may be considered as positive and his debts as negative. Thus $+300 - 350 = -50$ dollars. This -50 shows that the man is in debt 50 dollars, or has virtually 50 dollars less than nothing.

Again when a ship sails a miles to the north of the equator, it is sailing in a positive direction as compared with a course to the south of the equator. For example, if a ship travels 200 miles north it is said to have traversed $+200$ miles; on the other hand if it turns about and sails in the reverse direction 250 miles, it then will have a position 50 miles to the south of the starting point, *i.e.*, its position will be -50 miles. Expressing this in equation form gives $200 - 250 = -50$.

Examples. 1. What is the difference in longitude between two places where the longitudes are -80° and $+30^\circ$?

SOLUTION. Since one position is 80° from the starting point in one direction and the other 30° in the opposite direction from the starting point, the difference between the two places will be represented by the equation:

$$80 + 30 = 110^\circ \quad \text{Ans.}$$

2. A man has bills receivable to the amount of 500 dollars, and bills payable to the amount of 1,000 dollars; how much is he worth?

SOLUTION. Having bills receivable to the amount of \$500, this is what he is worth; but having bills payable to the amount of \$1,000, this is what he owes. He is actually worth then:

$$\begin{aligned} \$500 - \$1000 &= -\$500 \quad \text{Ans.} \\ \text{or he is in debt } &\$500 \end{aligned}$$

PROBLEMS FOR PRACTICE

1. The temperature at 6 P. M. is $+14^{\circ}$ and during the evening it grows colder at the rate of 4° an hour. Required the temperature at 9 P. M., at 10 P. M., and at midnight.

2. What is the difference in latitude between two places where the latitudes are $+86^{\circ}$ and -14° ?

69. Parenthesis. The parenthesis has already been shown to indicate that the terms enclosed are to be considered as one quantity. The following rules indicate the proper use of the parenthesis.

Rules. (a) *When a parenthesis is preceded by the $+$ sign the parenthesis may be removed without making any change in the expression within the parenthesis.*

(b) *When a parenthesis is preceded by the $-$ sign the parenthesis may be removed if the sign of every term within the parenthesis be changed.*

(c) *When a number or letter immediately precedes or follows the parenthesis, with no sign between, the multiplication sign is understood.*

(d) *When the first term within a parenthesis has no sign before it, the $+$ sign is always understood.*

Examples. 1. Remove the parenthesis from the expression: $8 + (2 + 6)$.

$$8 + (8) = 8 + 8 = 16$$

In this problem, 2 is added to 6 before the parenthesis is removed, and the result is added to 8 to give the total sum of 16. It should be noted that the parenthesis has been removed without changing the signs within the parenthesis.

2. Remove the parenthesis from $a + (b + c - d)$.

$$a + (b + c - d) = a + b + c - d$$

3. Remove the parenthesis from $12 - (6 - 4)$

According to the rule, since the quantity $(6 - 4)$ is preceded by a minus sign the removal of the parenthesis makes it necessary to change the $+ 6$ to $- 6$, and the $- 4$ to $+ 4$. The expression then becomes $12 - 6 + 4$, or 10.

If the operation indicated had been performed before removing the parenthesis and the result subtracted from 12 the remainder would have been the same.

4. Remove the parentheses from $c + k - (m - 1) + (a + b)$.

The signs of m and 1 are changed to $-$ and $+$ respectively on the

removal of the parentheses and the final expression becomes $c+k-m+1+a+b$.

5. Remove the parentheses from $a-6(b+7c-e)$.

Removing the parentheses and changing the necessary signs the expression becomes $a-6(b+7c-e)=a-6b-42c+6e$. Ans.

PROBLEMS FOR PRACTICE

Remove the parentheses in the following problems:

1. $6 - (4 + 10)$
2. $a + b - (c - ax + 5)$
3. $36 - (14 - 8) + (3 - 10)$
4. $c - (a + b) + (c^2 - m)$

EQUATIONS

70. An equation is an expression of equality between two quantities. It has been proved that the proportion $\frac{a}{b} = \frac{x}{d}$ can be expressed in the form $ad = bx$, by the law of the means and extremes. Since the products ad and bx are equal the whole expression, $ad = bx$, is called an *equation*. The quantities on the left side of the equality sign constitute the *first member*; those on the right side constitute the *second member*. Any quantity or group of quantities which is separated from others by the $+$ or $-$ sign is called a *term* of the equation.

The equation has the following properties:

(a) *If letters occur in an equation, they must be given such numerical values that when substituted in the equation both members will be numerically equal.* In the equation $2b = 4k$, let $b = 4$ and $k = 2$. Substituting these values, the equation becomes $2 \times 4 = 4 \times 2$, or $8 = 8$.

(b) *The same quantity may be added to or subtracted from each member without destroying the equality.* In the equation, $2b = 4k$, let 8 be added to each member. Thus, $2b + 8 = 4k + 8$. Then substituting the values given in (a) in the above equation, it becomes

$$\begin{aligned}(2 \times 4) + 8 &= (4 \times 2) + 8 \\ 8 + 8 &= 8 + 8 \\ 16 &= 16\end{aligned}$$

It is seen that the quantities within the parentheses, which represent the values of the members of the original equation, are equal and the

final result shows that the equality has not been destroyed by adding the same quantity to both members.

Similarly it can be shown that the same quantity may be subtracted from both members without destroying the equality.

(c) *Each member of the equation may be multiplied or divided by the same quantity without destroying the equality.* For example, divide by 3 each side of the equation $3x + 6c = 9m + 15$.

$$\frac{3x + 6c}{3} = \frac{9m + 15}{3}$$

Let $x = 25$, $c = 5$, and $m = 10$. Substituting these values in the equation, it becomes $\frac{(3 \times 25) + (6 \times 5)}{3} = \frac{(9 \times 10) + 15}{3}$

$$\begin{aligned}\frac{75 + 30}{3} &= \frac{90 + 15}{3} \\ \frac{105}{3} &= \frac{105}{3}\end{aligned}$$

It is seen that the equality has not been destroyed by the division. Similarly it may be shown that the same root of each member may be extracted, or that each member may be raised to the same power without destroying the equality.

(d) *When a term has no sign before it, the plus sign is understood.*

(e) *The order of the terms or of the letters within the terms is unimportant; the proper sign, however, must be given to each term.* For example $6abc + 4cy = 0$ is the same as $6cab + 4yc = 0$. Again, $3a - 6bc - xy = 40$ is equivalent to $-xy + 3a - 6bc = 40$.

71. Transposition. (a) Suppose it is required to solve for the unknown value of x in the equation $4x + 2 = 6 + 3x$. In this equation the unknown quantity is found in both members. *In any equation containing the unknown quantity in both members it is common to arrange the unknown values on the left side of the equation, and the known values on the right. After combining the terms divide both sides of the equation by the coefficient of the unknown quantity. The above operation may be accomplished by subtracting $3x$ and 2 from both members of the equation which results in the following:*

$$\begin{aligned}4x + 2 - 3x - 2 &= 6 + 3x - 3x - 2 \\ \text{combining terms} \quad x + 0 &= 4 + 0 \\ x &= 4, \text{ Ans.}\end{aligned}$$

Again, solve for x in the equation

$$8x + 4x + 6 = 2x + 8$$

Subtracting $2x$ and 6 ,

from both members, $8x + 4x + 6 - 2x - 6 = 2x + 8 - 2x - 6$
 combining terms

$$10x = 2$$

$$\text{or } x = .2 \text{ Ans.}$$

(b) The above form of solution may be simplified by the use of a method known as *transposition*. *All terms containing the unknown quantity, which are found on the right side, are transposed to the left, at the same time changing their signs; likewise all known terms found on the left side are transposed to the right, at the same time changing signs.* If, when the terms have been combined, x is found to be $-$, change its sign to $+$ and at the same time change the sign of its numerical equivalent. (See Examples 6 and 8 following.)

Examples. 1. Solve the equation, $4x + 2 = 6 + 3x$.

SOLUTION: Bring over the $3x$ to the left-hand side of the equation and change its sign from $+$ to $-$; also transpose the 2 to the right-hand side and change its sign to $-$; thus,

$$\text{Transposing} \quad 4x - 3x = 6 - 2$$

$$\text{Combining} \quad x = 4 \text{ Ans.}$$

2. Solve the equation: $8x + 4x + 6 = 2x + 8$

$$\text{Transposing} \quad 8x + 4x - 2x = 8 - 6$$

$$10x = 2$$

$$x = \frac{2}{10} \text{ or } .2 \text{ Ans.}$$

3. Solve for m in the equation, $10m + 12 - 2m = 24 - 2m$

$$\text{Transposing} \quad 10m - 2m + 2m = 24 - 12$$

$$\text{Combining} \quad 10m = 12$$

$$m = \frac{12}{10} = 1.2 \text{ Ans.}$$

4. Solve the equation: $9x = 6x + 20$

$$\text{Transposing} \quad 9x - 6x = 20$$

$$\text{Combining} \quad 3x = 20$$

$$x = \frac{20}{3} \text{ or } 6\frac{2}{3} \text{ Ans.}$$

5. Solve the equation:
- $4x-3=61$

Transposing $4x=61+3$

Combining $4x=64$

$$x = \frac{64}{4} \text{ or } 16 \quad \text{Ans.}$$

6. Solve the equation:
- $8x-22=12x-18$

Transposing $8x-12x=22-18$

Combining $-4x=4$

Dividing each side by 4 and changing signs,

$$x = -1 \quad \text{Ans.}$$

7. Solve the equation:
- $32x+24=30-50x$

Transposing $32x+50x=30-24$

Combining $82x=6$

$$x = \frac{6}{82} \text{ or } \frac{3}{41} \quad \text{Ans.}$$

8. Solve the equation:
- $12-13x=5-10x$

Transposing $-13x+10x=5-12$

Combining $-3x=-7$

Changing signs $3x=7$

$$x = \frac{7}{3} \text{ or } 2\frac{1}{3} \quad \text{Ans.}$$

PROBLEMS FOR PRACTICE

Solve for x in the following equations:

1. $5x+9=14-2x$

2. $6x-28=15x-16$ Ans. $-1\frac{1}{3}$

3. $20-14x=26-18x$

4. $12x+25-35=14x+22x-22$

5. $81+10-20x=60+40x$

72. Equations Containing Fractions. The equations which have so far been given illustrate more simple algebraic forms. In practice, equations may often contain fractions and it will be well to study the method of handling such equations.

In the treatment of fractions, Part II, it was shown that a group of fractions could be added or subtracted only by reducing them all to a common denominator. This same process may be applied to

an equation containing fractions. For example, let us take the equation

$$\frac{4x+7}{7} + \frac{x}{6} = 13$$

By inspection the L. C. D. is seen to be 42. Reducing each term of the equation to the least common denominator, we have

$$\frac{(6 \times 4x) + (6 \times 7)}{6 \times 7} + \frac{(7 \times x)}{7 \times 6} = \frac{42 \times 13}{42}$$

or

$$\frac{24x+42}{42} + \frac{7x}{42} = \frac{546}{42}$$

Combining

$$\frac{(24x+42+7x)}{42} = \frac{546}{42}$$

It has just been learned in article 70 that each member of an equation may be multiplied by the same quantity without destroying the equality. Multiplying the above equation by 42, we have

$$\frac{\cancel{42} (24x+42+7x)}{\cancel{42}} = \frac{\cancel{42} \times 546}{\cancel{42}}$$

Cancelling like terms in numerator and denominator of each fraction, we have

$$\begin{aligned} 24x+42+7x &= 546 \\ 31x &= 504 \\ x &= 16.2+ \end{aligned}$$

An inspection of the last fractional equation shows that multiplying by 42 was unnecessary as the same result would have been obtained by simply dropping the common denominator from each term. This leads to the following rules:

Rules. (a) *To solve an equation containing fractions, reduce all terms of the equation to a common denominator in the usual manner and then drop the denominators (This process is termed "clearing of fractions".) The resulting equation is then solved for the unknown quantity by the usual process.*

(b) *If fractions occur in the numerator or denominator of any term, this term must be reduced to a simple fraction before finding the L. C. D.*

Examples. 1. Solve the equation $\frac{6x}{8} - (7+x) = \frac{2}{3}$

Reducing to L. C. D. $\frac{18x}{24} - \frac{24(7+x)}{24} = \frac{16}{24}$

Dropping denominator $18x - 168 - 24x = 16$

$$-6x = 184$$

$$6x = -184$$

$$x = -30.6 + \text{ Ans}$$

2. Solve the equation $\frac{8x+3}{15} - \frac{(2-4x)}{5} = 3$

Clearing of fractions $8x+3-6+12x=45$

Combining $20x=48$

$$x=2.4 \text{ Ans.}$$

PROBLEMS FOR PRACTICE

Solve for x in the following equations:

$$1. \quad \frac{x}{20} - \frac{2}{4} = \frac{4}{5}$$

$$2. \quad \frac{(8x+3)}{12} + \frac{1}{2} = 8$$

$$\text{Ans. } 10\frac{7}{8}$$

$$3. \quad \frac{42-x}{25} - \frac{(2x+14)}{10} = 20$$

$$\text{Ans. } -82.16 +$$

73. General Application of Equations to Problems.

In the preceding problems the relation of the quantities has been expressed directly in equation form and solutions given for the value of the unknown quantity. However, the equation finds its greatest field of usefulness in problems in applied science and engineering where no equation is given directly but must be formed from conditions stated in the problem itself.

The solution of such a problem consists of two distinct parts: (1) The *statement* of the problem, and (2) the *solution* of the equation.

The statement of the problem is the process of expressing the conditions of the problem in the form of an equation. The statement of the problem is often more difficult to beginners than the solution of the equation. No rule can be given for the

statement of every particular problem. Much must depend on the skill of the student, and practice will give him readiness in this process. The following is the general plan of finding the equation.

(a) *Study the problem, to ascertain what quantities in it are known and what are unknown, and to understand it fully, so as to be able to prove the correctness or incorrectness of any proposed answer.*

(b) *Represent the unknown quantity by x , and express in equation form the relations which hold between the known and unknown quantities; an equation will thus be obtained which can be solved by the methods already given, and from which the value of the unknown quantity may be found.*

The problems which follow the illustrative examples will test the ability of the student to state the problem in equation form, and solve for the unknown quantity.

Examples. 1. A , B , and C receive \$1285 among them; A 's share is \$25 more than $\frac{5}{6}$ ths of B 's, and C 's is $\frac{4}{15}$ ths of B 's. Find the share of each.

In this problem we must first determine what unknown quantity is to equal x . Since both A 's and C 's shares are compared with B 's, (or depend upon B 's), we may let x represent B 's share. Now, A 's share is \$25 more than $\frac{5}{6}$ ths of B 's, and as B 's share is x , A 's must be $(\frac{5}{6} \times x) + 25$, or $\frac{5x}{6} + 25$. As C 's share is $\frac{4}{15}$ ths of B 's, it will be $\frac{4}{15} \times x$ or $\frac{4x}{15}$.

We now have a statement of all three shares, and since together the men have \$1285, the sum of their shares must equal \$1285. We, therefore, place the sum of the three shares as the first member of the equation, and the actual value of these three shares in dollars for the second member of the equation, as follows:

SOLUTION. Let x represent B 's share
 then $\frac{5x}{6} + 25$ represents A 's share
 and $\frac{4x}{15}$ represents C 's share

Forming the equation, we have:

$$x + \left(\frac{5x}{6} + 25 \right) + \frac{4x}{15} = 1285$$

Clearing equation of fractions:

$$30x + 25x + 750 + 8x = 38,550$$

Transposing:

$$30x + 25x + 8x = 38,550 - 750$$

Combining terms:

$$\begin{aligned} 63x &= 37,800 \\ x &= \$600, B's \text{ share} \end{aligned}$$

Since we have B 's share, to find the other two shares, we substitute \$600 for x , as follows:

$$A's \text{ share} = \frac{5x}{6} + 25 = \frac{(5 \times 600)}{6} + 25 = \$525$$

$$C's \text{ share} = \frac{4x}{15} = \frac{4 \times 600}{15} = \$160$$

$$\begin{array}{l} \text{Answers} \quad \left\{ \begin{array}{l} A's \text{ share } \$525 \\ B's \text{ share } \$600 \\ C's \text{ share } \$160 \end{array} \right. \end{array}$$

2. Divide 19 into two parts such that 7 times the less shall exceed 6 times the greater by 3.

SOLUTION. Let x = the lesser part

$(19 - x)$ = the greater part

then $7x = 7$ times the lesser part and $6(19 - x) = 6$ times the greater part, but 7 times the lesser part must exceed 6 times the greater by 3.

Therefore, $7x - 6(19 - x) = 3$

Expanding, $7x - 114 + 6x = 3$

Transposing, $13x = 114 + 3$

$$13x = 117$$

Dividing by 13, $x = \frac{117}{13} = 9$ Ans.

Therefore, $x = 9$, the lesser part; $19 - 9 = 10$, the greater part. It can readily be seen that the conditions have been met, as 7 times the lesser, 63, exceeds 6 times the greater, 60, by 3.

3. In an ingot of brass weighing 105 lb., if the copper used weighed 20 lb. less and the tin 20 lb. more, the weight of the copper would still be 5 lb. more than the weight of the tin. What is the weight of each?

SOLUTION. Let x represent weight of copper and $(105 - x)$ represent weight of tin.

Then by the conditions of the problem, the weight of the copper reduced by 20 lb., that is $(x-20)$, is 5 lb. more than the weight of tin increased by 20 lb., that is, $(105-x) + 20$. The equation is, therefore,

$$(x-20) - 5 = (105-x) + 20$$

Transposing $x + x = 105 + 20 + 20 + 5$

$$2x = 150$$

$$\left. \begin{array}{l} x = 75 \text{ lb. weight of copper} \\ 105 - 75 = 30 \text{ lb. weight of tin} \end{array} \right\} \text{Ans.}$$

PROBLEMS FOR PRACTICE

1. A bar of solder weighs 51 lb. It is composed of tin and lead. There are 9 lb. more tin in the bar than lead. How much lead is there? How much tin?

(Suggestion: Let x = the amount of lead. Then $51 - x$ = the amount of tin.)

Ans. 30 lb. tin. 21 lb. lead.

2. Divide a freight train of 76 cars into two trains so that twice the longer train shall be less by 16 than 4 times the shorter train.

(Suggestion: x = number of cars in longer train; $76 - x$ = number of cars in shorter train.)

Ans. 48, 28.

3. Two dynamos furnish 84 horsepower. One dynamo furnishes $\frac{3}{4}$ as much power as the other. What does each furnish?

(Suggestion: x = horsepower of one machine, $\frac{3x}{4}$ = horsepower of other machine.)

Ans. 48 h.p., 36 h.p.

4. In a ton of prepared mortar, the sand weighs 200 pounds more than the gypsum and the plaster of Paris one-fourth as much as the gypsum. What is the weight of each?

5. A tank can be filled by two pipes A and B , in 12 and 15 minutes respectively. A service pipe C will empty the tank in 10 minutes. How long will it take A and B to fill the tank when C is emptying it?

Ans. 20 minutes.

APPENDIX

MEASURES OF EXTENSION

Linear Measure

12 inches (in. or ")	= 1 foot (ft. or ')
3 feet	= 1 yard (yd.)
5½ yards or 16½ ft.	= 1 rod (rd.), pole, or perch
320 rods or 5280 ft.	= 1 mile (mi.)

A *knot*, used in navigation, is equal to 1.15 miles.

A *fathom* is equal to six feet.

Square Measure

144 square inches (sq. in.)	= 1 square foot (sq. ft.)
9 square feet	= 1 square yard (sq. yd.)
30½ square yards	= 1 square rod (sq. rd.)
160 square rods	= 1 acre (A.)
640 acres	= 1 square mile (sq. mi.)

Cubic Measure

1728 cubic inches (cu. in.)	= 1 cubic foot (cu. ft.)
27 cubic feet	= 1 cubic yard (cu. yd.)
128 cubic feet	= 1 cord (cd.)
8 cord feet	= 1 cord (cd.)

A *perch* of stone or masonry is 16½ feet (1 rod) long, 1½ feet wide, and 1 foot high, or 24½ cu. ft. capacity. The perch sometimes has other values, viz, 22 cu. ft. and 18 cu. ft. The value first mentioned is used throughout this text.

16 cu. ft. = 1 cord foot (cd. ft.). A pile of wood 8 feet long, 4 feet wide, and 4 feet high contains a cord. The cord foot is 1 foot of the length of such a pile.

Surveyors' Linear Measure

7.92 inches (in.)	= 1 link (l.)
25 links	= 1 rod (rd.)
100 links	} each = 1 chain (ch.)
4 rods	
66 feet	
80 chains	= 1 mile (mi.)

Surveyors' Square Measure

16 square rods	= 1 square chain
10 square chains	= 1 acre (A.)
640 acres	= 1 square mile (sq. mi.)
1 square mile	= 1 section (sec.)
36 sections	= 1 township (Tp.)

Surveyors' measure was obtained by calling one tenth of an acre a *square chain*. A tenth of an acre is equal to 16 square rods, and is equivalent to a square each side of which measures 4 rods. Thus, since 16 square rods is equal to one square chain, a linear chain is 4 rods or 792 inches. The linear chain is divided into 100 equal parts called *links*.

MEASURES OF CAPACITY**Dry Measure**

2 pints (pts.)	= 1 quart (qt.)
8 quarts	= 1 peck (pk.)
4 pecks	= 1 bushel (bu.)

The *Winchester bushel* contains 2,150.4 cubic inches and is used in measuring shelled grains. The *heaped bushel* of 2747.7 cubic inches is used for measuring apples, potatoes, corn in the ear, etc. The *dry gallon*, or half peck, contains 268.8 cubic inches.

Liquid Measure

4 gills (gi.)	= 1 pint (pt.)
2 pints	= 1 quart (qt.)
4 quarts	= 1 gallon (gal.)
31½ gallons	= 1 barrel (bbl.)
2 barrels, or 63 gallons	= 1 hogshead (hhd.)

One U. S. standard gallon contains 231 cubic inches, and one gallon of water weighs approximately 8½ pounds.

One cubic foot of water contains approximately 7.5 U. S. standard gallons and weighs 62.3 pounds.

Apothecaries' Fluid Measure

60 minims	= 1 fluid drachm
8 fluid drachms.	= 1 fluid ounce
16 fluid ounces	= 1 pint

MEASURES OF WEIGHT

Troy Weight

24 grains (gr.)	= 1 pennyweight (dwt. or pwt.)
20 pennyweights	= 1 ounce (oz.)
12 ounces	= 1 pound (lb.)

The standard Troy pound contains 5,760 grains and is identical with the Troy pound of Great Britain. Troy weight is used in weighing gold, silver, and jewels.

The *carat* (car.), a weight of about 3.2 grains Troy, is used in weighing diamonds and other precious stones. The term carat is used also to express the fineness of gold, and means $\frac{1}{24}$ part. For example, gold is said to be 18 carats fine when it contains 18 parts of pure gold and 6 parts of alloy or baser metal.

Avoirdupois Weight

16 ounces (oz.)	= 1 pound (lb.)
100 pounds (lbs.)	= 1 hundredweight (cwt.)
20 hundredweight, or 2,000 lbs.	= 1 ton (t. or T.)

The Avoirdupois pound contains 7,000 grains Troy.

There are 7.3 Apothecary drams in an Avoirdupois ounce.

25 pounds are sometimes called a *quarter*.

The ton of 2,000 pounds is sometimes called the *short ton*.

Long Ton Weight

16 ounces	= 1 pound (lb.)
112 pounds (lbs.)	= 1 hundredweight (cwt.)
20 cwt. or 2,240 lbs	= 1 ton (T. or t.)

NOTE. The long ton table is not to be used unless specially designated.

Apothecaries Weight

20 grains	= 1 scruple (sc.)
3 scruples	= 1 dram (dr.)
8 drams	= 1 ounce (oz.)
12 ounces	= 1 pound (lb.)

The pound, ounce, and grain of this weight are identical with those of Troy weight.

Drugs and medicines are bought and sold at wholesale by Avoirdupois weight.

METRIC SYSTEM

The fundamental unit of the metric system is the *meter*—the unit of length. From this the units of capacity—*liter*—and of weight—*gram*—were derived. All other units are the decimal sub-divisions or multiples of these. These three units are simply related; *e. g.*, * for all practical purposes one *cubic decimeter* equals one *liter* and one *liter* of water weighs one *kilogram*. The metric tables are formed by combining the words *meter*, *gram*, and *liter* with the six numerical prefixes, as in the following tables:

PREFIXES	MEANING			UNITS
<i>milli</i>	one thousandth	$\frac{1}{1000}$	001	<i>meter</i> for length
<i>centi-</i>	= one hundredth	$\frac{1}{100}$	01	
<i>deci-</i>	= one tenth	$\frac{1}{10}$	1	
Unit	= one		1	<i>gram</i> for weight or mass
<i>deka-</i>	= ten	10	10	<i>liter</i> for capacity
<i>hecto-</i>	= one hundred	100	100	
<i>kilo-</i>	= one thousand	1000	1000	

Units of Length

milli-meter	=	001	meter
centi-meter	=	01	meter
deci-meter	=	1	meter
METER	=	1	meter
deka-meter	=	10	meter
hecto-meter	=	100	meter
kilo-meter	=	1,000	meter

Where *miles* are used in England and the United States for measuring distances, the *kilometer* (1,000 meters) is used in metric countries. It is about 5 furlongs. There are about 1,600 meters in a statute mile, 20 meters in a chain, and 5 meters in a rod.

The *meter* is used for dry goods, merchandise, engineering construction, building, and other purposes where the *yard* and *foot* are used. The meter is about a tenth longer than the yard.

The *centimeter* and *millimeter* are used instead of the *inch* and its fractions in machine construction and similar work. The centimeter, as its name shows, is the hundredth of a meter. It is used in cabinet

* The term "*e. g.*" means "for example"

work, in expressing sizes of paper, books, and in many cases where the inch is used. The centimeter is about two-fifths of an inch and the millimeter about one twenty-fifth of an inch. The millimeter is divided for finer work into tenths, hundredths, and thousandths.

If a number of distances in millimeters, meters, and kilometers are to be added, reduction is unnecessary. They are added as dollars, dimes, and cents are now added. For example, "1,050.25 meters" is not read "1 kilometer, 5 dekameters, 2 decimeters, and 5 centimeters," but "one thousand fifty meters and twenty-five centimeters," just as "\$1,050.25" is read "one thousand fifty dollars and twenty-five cents."

Area

The table of areas is formed by squaring the length measures, as in our common system. For land measure 10 meters square is called an *are* (meaning *area*). The side of one *are* is about 33 feet. The *hectare* is 100 meters square, and, as its name indicates, is 100 ares, or about $2\frac{1}{2}$ acres. An *acre* is about 0.4 hectare. A standard United States *quarter section* contains almost exactly 64 hectares. A *square kilometer* contains 100 hectares.

For smaller measures of surface the *square meter* is used. The square meter is about 20 per cent larger than the *square yard*. For still smaller surfaces the *square centimeter* is used. A *square inch* contains about $6\frac{1}{4}$ square centimeters.

Volume

The cubic measures are the cubes of the linear units. The *cubic meter* (sometimes called the *stere*, meaning *solid*) is the unit of volume. A *cubic meter* of water weighs a *metric ton* and is equal to 1 *kiloliter*. The *cubic meter* is used in place of the cubic yard and is about 30 per cent larger. This is used for "cuts and fills" in grading land, measuring timber, expressing contents of tanks and reservoirs, flow of rivers, dimensions of stone, tonnage of ships, and other places where the cubic yard and foot are used. The thousandth part of the cubic meter (1 cubic decimeter) is called the liter. (See table of capacity units.)

For very small volumes the *cubic centimeter* (c.c. or cm^3) is used. This volume of water weighs a *gram*, which is the unit of weight or mass. There are about 16 cubic centimeters in a cubic inch. The

cubic centimeter is the unit of volume used by chemists as well as in pharmacy, medicine, surgery, and other technical work. One thousand cubic centimeters make 1 liter.

Units of Capacity

milli-liter	=	.001	liter
centi-liter	=	.01	liter
deci-liter	=	1	liter
*LITER	=	1	liter
deka-liter	=	10	liter
hecto-liter	=	100	liter
kilo-liter	=	1,000	liter

The *hectoliter* (100 liters) serves the same purpose as the United States *bushel* (2,150.42 cubic inches), and is equal to about 3 bushels, or a barrel. A *peck* is about 9 liters. The liter is used for measurements commonly given in the *gallon*, the liquid and dry *quarts*, a liter being 5 per cent larger than our liquid quart and 10 per cent smaller than the dry quart. A *liter* of water weighs exactly a *kilogram*, i. e., 1,000 grams. A thousand liters of water weigh 1 metric ton.

Units of Weight (or Mass)

milli-gram	=	0.001	gram
centi-gram	=	.01	gram
deci-gram	=	.1	gram
GRAM	=	1	gram
deka-gram	=	10	gram
hecto-gram	=	100	gram
†kilo-gram	=	1,000	gram

Measurements commonly expressed in *gross tons* or *short tons* are stated in *metric tons* (1,000 kilograms). The metric ton comes between our long and short tons and serves the purpose of both. The *kilogram* and *half kilo* serve for everyday trade, the latter being 10 per cent larger than the pound. The kilogram is approximately 2.2 pounds. The *gram* and its multiples and divisions are used for the same purposes as ounces, pennyweights, drams, scruples, and grains. For foreign postage, 30 grams is the legal equivalent of the avoirdupois ounce.

*One liter equals 1.05668 liquid quarts or 0.9081 dry quarts.

†One kilogram equals 2.204622 avoirdupois pounds.

MEASURES OF TIME

60 seconds (sec.)	= 1 minute (min.)
60 minutes	= 1 hour (hr.)
24 hours	= 1 day (d.)
7 days	= 1 week (w. or wk.)
365 days or 52 wks.	= 1 year (yr.)
12 months (mos.)	= 1 year
366 days	= 1 leap year
100 years (yrs.)	= 1 century

In most business transactions 30 days are considered a month and 360 days a year.

The solar year is exactly 365 d. 5 hrs. 48 min. 49.7 sec.

A common year consists of 365 days for three successive years. Every fourth year, except as noted below, one day is added for the excess of the solar year over 365 days, and we have 366 days, which make what is called a *leap year*. The extra day is added to the month of February, which then has 29 days.

The following rule for leap year will make the calendar correct to within one day for a period of 4,000 years.

Every year exactly divisible by 4 is a leap year, the centennial years excepted; the other years are common years. Every centennial year exactly divisible by 400 is a leap year; the other centennial years are common years. Thus 1904 is a leap year, but 1905 is a common year; also, the year 2000 is a leap year, but 1800 and 1900 are common years.

MEASURES OF MONEY

United States Money

10 mills (mi.)	= 1 cent (¢)
10 cents	= 1 dime (d.)
10 dimes	= 1 dollar (\$)
10 dollars	= 1 eagle

These values, with the exception of the eagle, apply also to Canadian money.

English Money

4 farthings (far.)	= 1 penny (d)
12 pence	= 1 shilling (s)
20 shillings	= 1 pound (£) or sovereign

The unit of English money is the pound sterling, the value of which in United States money is \$4.8665.

French Money

1 franc = \$0.193

German Money

100 pfennige = 1 mark (Rm.) = \$0.238

MISCELLANEOUS MEASURES**Measures of Angles and Arcs**

60 seconds (")	= 1 minute (')
60 minutes	= 1 degree (°)
90 degrees	= 1 right angle or quadrant
360 degrees	= 1 circle

Stationers' Table

24 sheets	= 1 quire (qr.)
20 quires	= 1 ream (rm.)
2 reams	= 1 bundle (bdl.)
5 bundles	= 1 bale (bl.)

Comparative table of Metric and English equivalents.*

Linear Measure

1 inch	= 2.5 cm.
1 foot	= .3 meter
1 meter	= 39.37 inches
1 yard	= .9 meter
1 meter	= 1.1 yards
1 rod	= 5. meters
1 mile	= 1.6 kilometers
1 kilo	= .62 miles

Solid Measure

1 cu. inch	= 16.4 cu. cm.
1 cu. cm.	= .06 cu. in.
1 cu. foot	= 28.3 cu. dm.
1 cu. dm.	= .035 cu. feet
1 cu. yard	= .76 cu. meter
1 cu. meter	= 35.3 cu. feet
1 cord	= 3.6 staves

Square Measure

1 sq. inch	= 6.45 sq. cm.
1 sq. cm.	= .155 sq. in.
1 sq. foot	= .09 sq. meter
1 sq. meter	= 10.73 sq. feet
1 sq. yard	= .84 sq. meter
1 sq. rod	= 25.3 sq. meters
1 acre	= 40.5 ares
1 are	= .024 acres

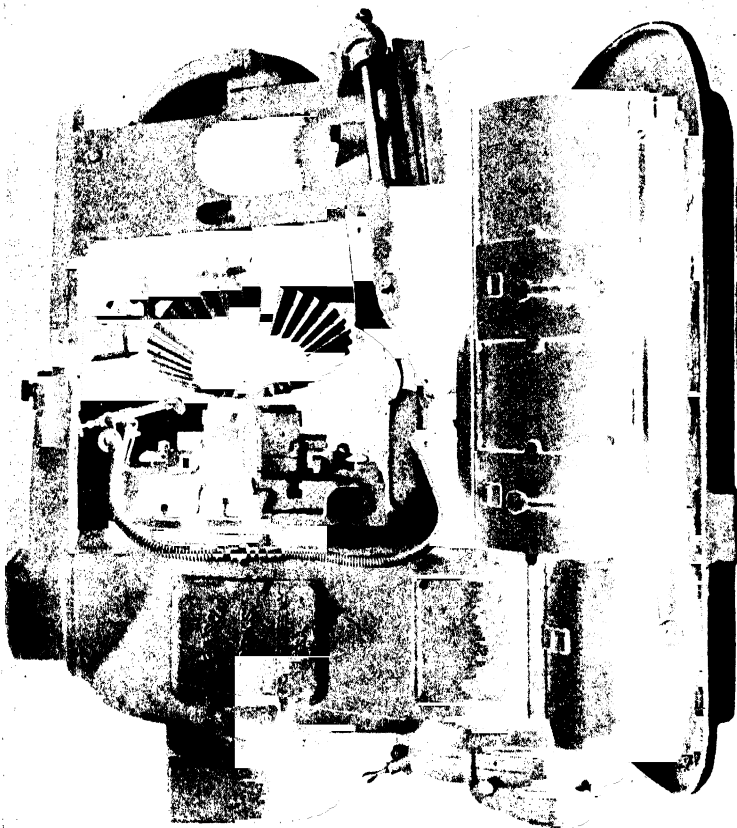
Measure of Capacity

1 fl. ounce	= .03 liter
1 liq. qt.	= .95 liter
1 liter	= 1.05 liq. qt.
1 gallon	= 3.8 liters
1 dry qt.	= 1.1 liters
1 liter	= .9 dry qts
1 bushel	= .35 hectoliter
1 hectoliter	= 2.84 bushels

Measures of Weight

1 grain Troy	= .06 gram	1 ounce Avoir.	= 28.35 grams
1 gram	= 15.4 grains Troy	1 gram	= .035 oz. Avoir
1 ounce Troy	= 31. grams	1 lb. Avoir.	= .45 kilogram
1 lb. Troy	= .37 kilogram	1 kilo	= 2.2 lbs. Avoir
1 kilo	= 2.7 lbs. Troy	1 metric ton	= 2200. lbs. Avoir

* Values given in these tables are only approximate but are sufficiently accurate for all ordinary computations. These tables need not be memorized.



25-INCH AUTOMATIC BEVEL GEAR PLANNER

Practical Mathematics

PART IV

INTRODUCTION

Mathematics has a language of its own and certain signs and symbols that belong to it. It is necessary for the student to know this language, and these signs and symbols and their uses, in order to understand his textbooks in mathematics. Failure to learn the exact meaning and use of the signs and symbols, and the definitions and terms used, keeps many students from mastering mathematical subjects.

It is sufficient at this time to call your attention briefly to those signs with which you are already familiar and bespeak a thorough mastery of the subject matter in this textbook.

Review of Signs. The following signs indicate operations to be performed:

Addition	+
Subtraction	-
Multiplication	$\times$
Division	$\div$

The signs denoting operations are used thus: $4+2=6$; $6-2=4$; $4\times 2=8$; $8\div 2=4$. They are not used when the numbers are written one below the other, arranged for addition, subtraction, or multiplication, or when the dividend is written with the divisor at the left, with a curved line between, arranged for division.

It will be remembered also that a coefficient (Part I, page 15) indicates multiplication, and that the line of a fraction indicates division. Thus, $7a$ means 7 times a , and $\frac{1}{2}(B\times A)$ means $\frac{1}{2}$ times B times A ; $\frac{1}{3}$ means 11 divided by 3.

The equality sign $=$ is used to show that two quantities or expressions are equal. It is never used except when the quantities so joined *are* equal.

The parentheses $()$ and the vinculum $\overline{\hspace{1cm}}$ are used to show that the quantities so tied together are to be treated as one.

Order of Signs. In a series of operations denoted by the signs of addition, subtraction, multiplication, and division, the multiplications must be performed first, the divisions next, and lastly the additions and subtractions.

If several additions or several multiplications occur together, they may be performed in any order.

If several subtractions or several divisions occur together, they *must* be performed in the order in which they come from left to right.

The work of solving problems may often be simplified by using cancellation. (See Part I, page 26.) It must be remembered that this method cannot be used when there are additions or subtractions indicated in the problem, because it is only common *factors* that are cancelled. When you cancel a factor, you have divided by that factor.

$$\frac{4 \times 9}{2} = \frac{\overset{2}{\cancel{4}} \times 9}{\underset{2}{\cancel{2}}} = 18$$

but $\frac{4+9}{2} = \frac{13}{2}$ or $6\frac{1}{2}$

and $\frac{12-3}{3} = \frac{9}{3}$ or 3

The radical sign $\sqrt{\quad}$ is used to indicate that a root is to be extracted. (See Part III, page 76.) The index shows what root is to be taken. $\sqrt[3]{\quad}$ indicates the cube root. $\sqrt{\quad}$ indicates the square root. When no index appears the square root is understood.

A small number written at the right and slightly above a number indicates that a power of that number is to be found. (See Part III, page 74.) Thus 12^2 means that 12 is to be used twice as a factor, or that the second power is to be found. $12^2 = 12 \times 12 = 144$. The number indicating the power to be found is called the exponent.

MENSURATION

Mensuration is the process of computing the length of lines, the area of surfaces, and the volume of solids.

Lines are measured in linear units. Areas are measured in square units. Volume is measured in cubic units.

A *line* has length only.

A *surface* has length and breadth (or width).

A *solid* has length, breadth, and thickness (or height).

These facts may be expressed in another way:

A line extends in one direction only.

A surface extends in two directions.

A solid extends in three directions.

These *directions of extension* are spoken of as dimensions.

We think of a point as something that has position only, without length, breadth, or thickness. Thus, the end of a line is a point. The intersection of two lines is a point.

LINES

74. A *straight line* is one that has the same direction throughout its length. It is the shortest distance between two points.

A *curved line* is one that is continually changing its direction. It is sometimes called a *curve*.

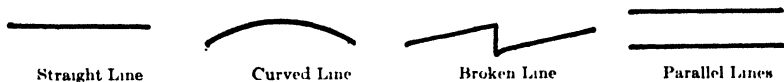
A *broken line* is one made up of several straight lines.

A *plane surface* (or simply a plane) is a surface such that when a straight edge is applied to it, the straight edge will in every part touch the surface.

If two straight lines in the same plane are extended, they will meet or they will not meet. If they ~~meet~~ they form angles. If they do not meet they are parallel.

Parallel lines are lines that lie in the same plane and are equally distant from each other at all points.

Lines are lettered to distinguish them; thus, if one end is marked *A* and the other *B*, it is called "the line *AB*," or "the line *BA*."



ANGLES

75. An *angle* is the measure of the difference in direction of two straight lines that meet. The lines are called *sides*, and the point of meeting, the *vertex*.

If one straight line meets another so that the angles formed are equal, the lines are said to be *perpendicular* to each other and the angles are *right angles*, Fig. 9.

An *acute angle* is less than a right angle, Fig. 10.

An *obtuse angle* is greater than a right angle and less than two right angles, Fig. 11.

Angles are distinguished by placing letters at the ends of the lines forming the sides and at the vertex. For example, in Fig. 10 the angle would be read as "the angle $A B C$ " or "the angle $C B A$." In writing, the sign $\angle$ is often used in place of the word "angle."

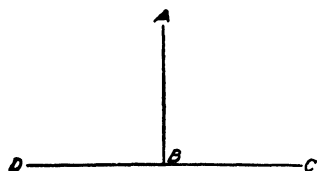


Fig. 9. Right Angle.

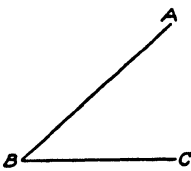


Fig. 10. Acute Angle.

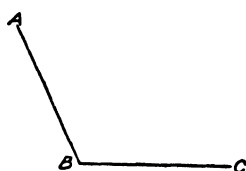


Fig. 11. Obtuse Angle.

As two right angles are formed when one line meets another perpendicularly, Fig. 9, it follows that:

(a) *The sum of all the angles about a point on one side of a straight line is equal to two right angles.*

(b) *The sum of all the angles about any point is equal to four right angles.*

When two lines intersect they form four angles, Fig. 12. The angles $A O C$ and $A O D$ are called *adjacent* angles because they have the same vertex O and the line $A O$ is common to both as it forms a side of each angle. It has been proved that $\angle A O C + \angle A O D =$ two

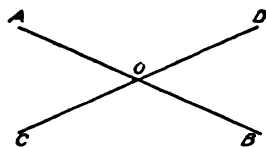


Fig. 12. Intersecting Lines.

right angles and it may also be proved that the angles $A O C$ and $D O B$, called *opposite* angles, are equal to each other. Therefore, *when two straight lines intersect, the opposite angles formed are equal to each other.*

Take two narrow strips of cardboard or stiff paper and lay them on the table crossing each other. Fasten at the point of crossing with a pin, so that the upper strip will turn freely. Set it first perpendicularly. You will see that you have made four right angles. Turn it in either direction. You will see that, no matter at what angle the one strip crosses the other, the sum of the angles formed will equal four right angles.

76. Measurement of Angles. It has been shown in Fig. 12 that the sum of the angles about the point of intersection of two lines is always four right angles. Evidently, other lines might be drawn through the same intersection, making the angles smaller and more in number, without changing the value of their sum. In measuring angles, therefore, it has been agreed to divide these four right angles about a point into 360 equal parts called *degrees*, indicated by the sign, $^{\circ}$. In this case each of the four right angles would contain

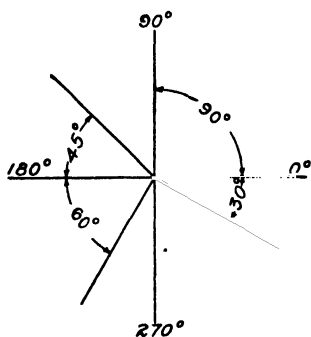


Fig. 13

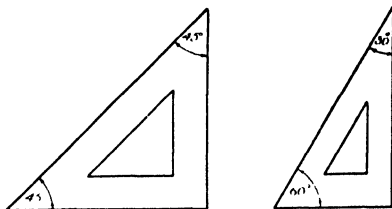


Fig. 13a.

90° while one-half, two-thirds, and one-third of a right angle would represent angles of respectively 45° , 60° , and 30° , as shown in Fig. 13. In drawing work, it will be found that the 45° triangle and the 30° — 60° triangle, Fig. 13a, are useful articles of equipment, but when it comes to accurate values of angles, some device like a *protractor*, Fig. 14, showing degrees and fractions of degrees, is necessary. Each degree is divided into 60 equal parts called *minutes* ($'$), and each minute into 60 equal parts called *seconds* ($''$). Thus to represent an angle of 23 degrees, 47 minutes, and 9 seconds, write $23^{\circ} 47' 9''$.

The relations of these units may be summarized as follows:

60 seconds (")	= 1 minute (').
60 minutes	= 1 degree (°).
360 degrees	= 1 circle, or circumference.
90 degrees	= 1 right angle.
180 degrees	= 2 right angles.
360 degrees	= 4 right angles, or 1 circumference.

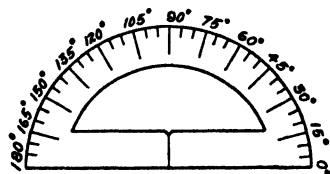


Fig 14 Protractor.

Two angles are *complementary* when their sum is equal to one right angle or 90° ; they are *supplementary* when their sum is equal to two right angles or 180° . Thus, an angle of 33° is the complement of one of 57° because $33 + 57 = 90$. Angles of 148° and 32° are supplementary because $148 + 32 = 180$.

PROBLEMS FOR PRACTICE

- How many seconds in 180° ? Ans. 648,000
- How many right angles in $202^\circ 30'$? Ans. $2\frac{1}{4}$
- What is the complement of 37° ?
- What is the complement of $11^\circ 47' 3''$? Ans. $78^\circ 12' 57''$
- What is the supplement of $131^\circ 4' 27''$? Ans. $48^\circ 55' 33''$
- Does an acute angle contain more or less than 90° ? an obtuse angle?
- In Fig. 12, the angle AOC is, say, 52° ; find the value of the three other angles about the point O .
- How many seconds in $30'$?

PLANE FIGURES

A *plane figure* is a plane surface bounded by lines, either straight or curved.

The distance around a plane figure is called its *perimeter*.

The *area* of a plane figure is the number of square units it contains.

POLYGONS

77. A *polygon* is a plane figure bounded by straight lines. The boundary lines are called the *sides* and the sum of the sides is called the *perimeter*.

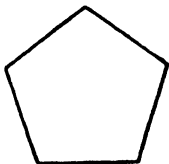


Fig. 15. Pentagon.

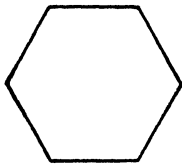


Fig. 16. Hexagon.

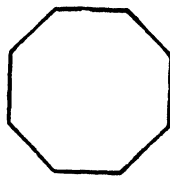


Fig. 17. Octagon.

Polygons are classified according to the number of sides.

A *triangle* is a polygon of three sides.

A *quadrilateral* is a polygon of four sides.

A *pentagon* is a polygon of five sides, Fig. 15.

A *hexagon* is a polygon of six sides, Fig. 16.

A *octagon* is a polygon of eight sides, Fig. 17.

An *equilateral polygon* is one all of whose sides are equal.

An *equiangular polygon* is one all of whose angles are equal.

A *regular polygon* is one all of whose angles and all of whose sides are equal.

TRIANGLES

78. A *triangle* is a polygon enclosed by three straight lines called *sides*. The *angles* of a triangle are the angles formed by the sides.

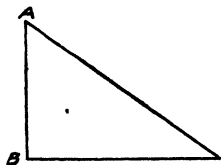


Fig. 18. Right-Angled Triangle

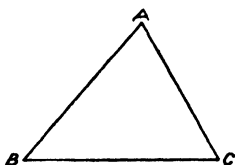


Fig. 19. Acute-Angled Triangle

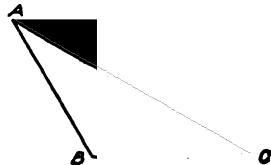


Fig. 20. Obtuse-Angled Triangle

A *right-angled* triangle, often called a *right* triangle, Fig. 18, is one that has a right angle. The longest side (the one opposite the

right angle) is called the *hypotenuse* and the other sides are sometimes called *legs*.

An *acute-angled* triangle, Fig. 19, is one that has all of its angles acute.

An *obtuse-angled* triangle, Fig. 20, is one that has an obtuse angle.

An *equilateral* triangle, Fig. 21, is one having all of its sides equal.

An *equiangular* triangle is one having all of its angles equal.

An *isosceles* triangle, Fig. 22, is one two of whose sides are equal.

A *scalene* triangle, Fig. 23, is one no two of whose sides are equal.

(a) The *base* of a triangle is the lowest side; it is the side upon which the triangle is supposed to stand. Any side may, however, be taken as the base. In an isosceles triangle, the side which is not one of the equal sides is usually considered as the base.

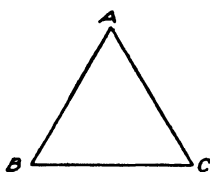


Fig. 21.
Equilateral Triangle.

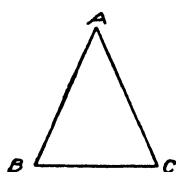


Fig. 22.
Isosceles Triangle.

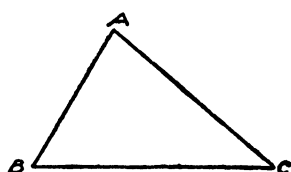


Fig. 23.
Scalene Triangle

(b) The *altitude* of a triangle is the perpendicular drawn from the vertex to the base AD , Fig. 24. In some triangles, Fig. 25, it

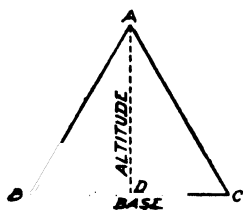


Fig. 24.

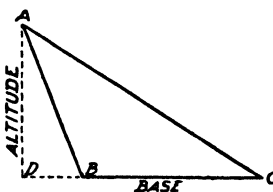


Fig. 25.

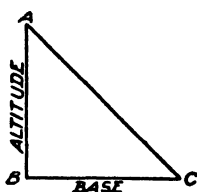


Fig. 26.

is necessary to produce the base so that the altitude may meet it. In a right triangle one leg may be considered as the base and the other as the altitude, Fig. 26.

Rule: *The sum of the angles of any triangle is equal to two right angles.*

Cut a triangle from a piece of paper or cardboard. Number the angles 1, 2, and 3. Cut off two points of the triangle. Fit the three angles together and you will see that they make the equivalent of two right angles, or 180° .

Keep this rule in mind, for you will find it useful.

Since you know that the sum of all the angles of a triangle is two right angles, or 180° , when you know two of the angles of a triangle you can readily compute the third angle.

Example. Given a right triangle with one of its angles $38^\circ 40'$, to find the third angle.

Solution. You know that one angle is 90° , since this is a right triangle, hence the sum of the other two angles must be 90° .

$$\begin{array}{r} 90^\circ - 38^\circ 40' = 51^\circ 20' \\ 90^\circ = 89^\circ 60' \\ \quad 38^\circ 40' \\ \hline \quad 51^\circ 20' \end{array}$$

PROBLEMS FOR PRACTICE

The following numbers in each case represent two angles of a triangle. Find the size of the third angle.

- | | |
|-----------------------------------|---------------------|
| 1. 90° and 45° | Ans. 45° |
| 2. 90° and 60° | Ans. 30° |
| 3. 120° and 40° | Ans. 20° |
| 4. $100^\circ 30'$ and 30° | Ans. $49^\circ 30'$ |
| 5. 60° and 60° | Ans. 60° |

How would you describe the triangle of Problem 5?

Ans. Equiangular and equilateral

(c) *Law of the right triangle.* If in the right triangle ABC , Fig. 27, the side AB is made 3 inches long, and the side AC is made 4 inches long, then the side BC , called the *hypotenuse*, is found to be 5 inches long.

The proof of this may be found by considering the accompanying figure. The square $ABED$ constructed on the side AB contains

9 squares; the square $L C A M$ constructed on the side $A C$ contains 16 squares, and the square $C F G B$ constructed on the hypotenuse $B C$ contains 25 squares. Now,

$$\begin{array}{r} 9 + 16 = 25 \\ 3^2 + 4^2 = 5^2 \\ \hline A B^2 + A C^2 = B C^2 \end{array}$$

Rule: *The square of the hypotenuse is equal to the sum of the squares of the other two sides.* By means of this relation it is possible

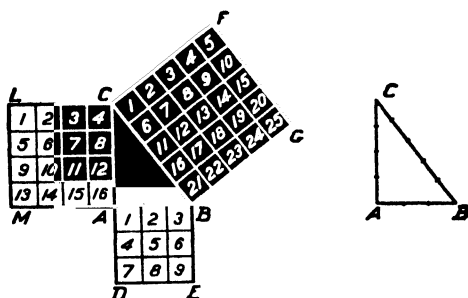


Fig 27 Graphical Proof of the Law of the Right Triangle.

to find the length of one side of a right triangle if the other two are known.

Examples. 1. A right triangle has sides 6 inches and 10 inches long respectively; what is the length of the hypotenuse?

Solution. Let x represent the hypotenuse.

$$\begin{aligned} x^2 &= 6^2 + 10^2 \\ &= 36 + 100 = 136 \\ x &= \sqrt{136} = 11.66 + \text{inches.} \quad \text{Ans.} \end{aligned}$$

2. The hypotenuse of a right triangle is 24 feet long and one of the short sides is 9 feet. How long is the other side?

Solution. Let x represent the other side.

$$\begin{aligned} 24^2 &= 9^2 + x^2 \\ x^2 &= 24^2 - 9^2 \\ &= 576 - 81 = 495 \\ x &= \sqrt{495} = 22.248 + \text{feet.} \quad \text{Ans.} \end{aligned}$$

3. A ladder 40 feet long when placed on the ground 24 feet from the bottom of a wall reaches to the top. Determine the height of the wall.

Solution. As the ladder is 40 feet long and the base is 24 feet from the wall, there is a right-angled triangle whose hypotenuse is 40 feet, one short side 24 feet, and the other short side to be determined. As in the previous example make use of the law of the right triangle and this will give $40^2 - 24^2 = 1024$. Taking the square root of 1024 will give 32 feet as the height of the wall.

If H = the hypotenuse and a and b the other sides, we have the formula

$$H^2 = a^2 + b^2$$

NOTE. The square is found by multiplying a number by itself. Hence a square is the product of two equal factors, $5 \times 5 = 25$; $5^2 = 25$. One of the two equal factors of a number is called its *square root*. The square root of 25 is 5, written $\sqrt{25} = 5$.

You already know the squares of numbers up to 12. The following squares will be useful for reference and will soon be learned by using them:

$13^2 = 169$	$20^2 = 400$
$14^2 = 196$	$21^2 = 441$
$15^2 = 225$	$22^2 = 484$
$16^2 = 256$	$23^2 = 529$
$17^2 = 289$	$24^2 = 576$
$18^2 = 324$	$25^2 = 625$
$19^2 = 361$	

PROBLEMS FOR PRACTICE

Draw figures and find the unknown side in each of the four problems following:

	Base	Perpendicular	Hypotenuse
1.	12 ft.	9 ft.	?
2.	?	18 ft.	30 ft.
3.	5 ft.	12 ft.	?
4.	3 yd.	?	5 yd.

5. A park is 40 rods long and 30 rods wide. What is the length of a walk running through it from opposite corners in a straight line?
Ans. 50 rods

6. The base and the altitude of a right triangle are equal, and the hypotenuse is 10 feet. Find the other two sides.

Ans. Each is 7.07+ft.

7. The foot of a ladder is 15 feet from the base of a building, and the top reaches a window 36 feet above the base. What is the length of the ladder?

Ans. 39 ft.

8. If the gable end of a house 24 feet wide is 10 feet high, what is the length of the rafters?

Ans. 15.62+ft.

9. A ladder 35 feet long touches a point in a wall 28 feet from the ground. How far from the base of the wall is the foot of the ladder?

Ans. 21 ft.

79. Areas of Triangles. A triangle is a surface and as such has two dimensions, *base* and *altitude*. Under *denominate numbers* the fact was brought out that surfaces or areas were measured by square measure, and that the area of the surface was proportional to the product of its two dimensions. If the surface is a triangle it has been found that the *area is equal to one-half of the product of the base by the altitude*. For example, in Fig. 24 and Fig. 25, the area of each triangle is $\frac{1}{2} (BC \times AD)$. Other methods might be given but, by the above method, most cases which are met may be easily solved.

Examples. 1. Find the area of the triangle $\triangle ABC$ shown in Fig. 26 if the base BC is 10 inches and the altitude AB is 10 inches.

Solution. Area = $\frac{1}{2} (\triangle ABC \times BC) = \frac{1}{2} (10 \times 10) = 50$ square inches.

2. Find the area of a right triangle whose base is 50 feet and the hypotenuse 200 feet.

Solution. To obtain the altitude, make use of the law of the right triangle, that is, take the square root of the difference in the squares of the hypotenuse and base of the triangle.

$$200^2 - 50^2 = 40000 - 2500 = 37500$$

$$\sqrt{37500} = 193.64 + \text{ft. altitude of the triangle.}$$

$$\text{Area of the triangle} = \frac{1}{2} (50 \times 193.64) = 4841 \text{ sq. ft.}$$

Since A = area, b = base, and a = altitude, we have the formula

$$A = \frac{b \times a}{2}$$

Knowing the rule for finding the area of a triangle it is easy to find one dimension when the area and one dimension are given.

Since
$$A = \frac{b \times a}{2}$$

$$2A = b \times a$$

Hence
$$\frac{2A}{b} = a$$

or
$$\frac{2A}{a} = b$$

Writing this out in full, you know that the area is one-half the product of the base and the altitude; therefore, base $\times$ altitude = $2 \times$ area; from this you easily deduce that $2 \times$ area, divided by either dimension will give the other dimension, because the product divided by either factor will give the other factor.

Examples. 1. The area of a triangle is 12 square feet; its altitude is 4 feet. Find its base.

$$A = \frac{(b \times a)}{2}$$

$$12 = \frac{(b \times 4)}{2}$$

$$2 \times 12 = b \times 4$$

$$\frac{24}{4} = b, \text{ or } b = 6$$

Hence the base is 6 feet.

Prove this by using the formula for area, and the values for base and altitude.

$$\text{Area} = \frac{1}{2} (6 \times 4) = 12 \text{ sq. ft.}$$

2. The area of a triangle is 56 square feet; its base is 14 feet. Find its altitude.

$$56 = \frac{1}{2} (14 \times a)$$

$$112 = 14 \times a$$

$$a = \frac{112}{14} = 8$$

Hence the altitude is 8 feet.

Proof:

$$\frac{1}{2}(8 \times 14) = 56$$

In finding the area of a right triangle, remember that the two short sides are perpendicular to each other and that, therefore, the area is equal to one-half the product of the two short sides.

PROBLEMS FOR PRACTICE

1. A piece of sheet metal in the form of a triangle weighs 18 pounds. If the height of the triangle is 8 feet and the base 3 feet, what is the weight of the metal per square foot? Ans. $1\frac{1}{2}$ lb.

2. In surveying a triangular piece of land it was found that two of the angles measured $44^\circ 56' 4''$ and $31^\circ 11' 8''$. Compute the value of the third angle. Ans. $103^\circ 52' 48''$

3. An iron brace used in supporting a shelf is fastened to the wall 18 inches below the shelf and to the shelf 12 inches from the wall. Find the length of the brace. Ans. $21.63 + \text{in.}$

4. The area of a triangle is 24 square inches. If the altitude is 6 inches, find the length of the base. Ans. 8 in.

5. An iron chimney is supported by a guy wire which makes an angle of $63^\circ 24'$ with the ground. Determine the angle between the chimney and guy wire. Ans. $26^\circ 36'$

6. Find the altitude of a triangle whose area is 48 acres and whose base is 48 rods. Ans. 1 mi.

7. If one end of a ladder 50 feet long rests on the ground 10 feet from the base of a wall, how far from the ground does the top of the ladder touch the wall? Ans. 49 — ft.

8. Find the area of the right triangle of Problem 7.
 9. Find the area of a triangle whose base is 9 inches and altitude 6 inches. Ans. 27 sq. in.

QUADRILATERALS

80. A *quadrilateral* is a polygon bounded by four straight lines, as Fig. 28.

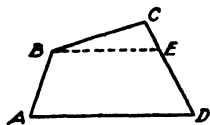


Fig. 28. Quadrilateral.

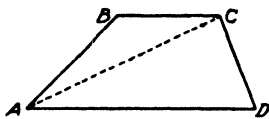


Fig. 29. Trapezoid.

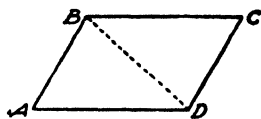


Fig. 30. Parallelogram.

A *trapezoid* is a quadrilateral having two sides parallel, Fig. 29. The parallel sides are called the *bases* and the perpendicular distance between the bases is called the *altitude*.

A *parallelogram* is a quadrilateral whose opposite sides are parallel, Fig. 30.

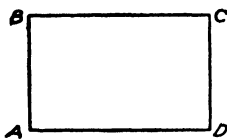


Fig. 31. Rectangle.

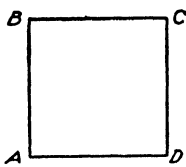


Fig. 32. Square

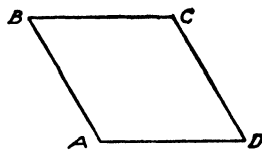


Fig. 33. Rhombus.

The three important kinds of parallelograms are as follows. The *rectangle*, Fig. 31, whose angles are right angles.

The *square*, Fig. 32, all of whose sides are equal and whose angles are right angles.

The *rhombus*, Fig. 33, whose sides are equal but whose angles are not right angles.

81. **Areas of Quadrilaterals.** Considering first the *trapezoid*, it will be seen that in Fig. 29 the dotted line AC (called a *diagonal*) divides the figure into two triangles, ABC and ACD , whose altitudes are the same, namely, the altitude of the trapezoid. By a previous rule the areas of these triangles will be respectively

$$\frac{1}{2} (\text{altitude} \times BC) \text{ and } \frac{1}{2} (\text{altitude} \times AD)$$

Therefore Area of trapezoid = altitude $\times \frac{B C + A D}{2}$

Rule: (a) *The area of a trapezoid is equal to the product of the altitude by one-half the sum of the bases.*

The area of a *quadrilateral* like that shown in Fig. 28 may be obtained by drawing a line, $B E$, parallel to the base $A D$. The figure will then be divided into a trapezoid and an extra triangle whose areas are measurable by the methods given.

The *parallelogram*, Fig. 30, may also be divided by means of a diagonal into two triangles whose areas are respectively $\frac{A D \times \text{altitude}}{2}$

and $\frac{B C \times \text{altitude}}{2}$. But $A D$ and $B C$ are equal and, consequently,

the sum of these areas = $\frac{2(A D \times \text{altitude})}{2} = A D \times \text{altitude}$.

Example. Find the area of a trapezoid whose parallel sides are 60 rods and 80 rods, and whose altitude is 30 rods.

Solution. Represent the two bases by b and b' . The formula may then be written

$$A = \frac{b + b'}{2} \times a$$

Substituting the values given for this problem

$$\text{Area} = \frac{60 + 80}{2} \times 30$$

$$\text{Area} = 70 \times 30 = 2100$$

Since

$$160 \text{ sq. rd.} = 1 \text{ Acre}$$

$$2100 \text{ sq. rd.} = 13 \text{ A. } 20 \text{ sq. rd.}$$

Rule: (b) *The area of a parallelogram is equal to the product of the base and the altitude.* This will be true of any parallelogram and hence true of the rectangle, Fig. 31, the square, Fig. 32, and the rhombus, Fig. 33. In the case of the first two, the fact that the angles are all right angles makes the length of the vertical side become the altitude.

$$A = b \times a$$

Find the area in acres of a parallelogram whose base is 80 rods and altitude 60 rods.

$$A = b \times a$$

$$A = 80 \times 60 = 4800$$

$$4800 \text{ sq. rd.} = 30 \text{ acres}$$

The complete solution may be written

$$A = \frac{80 \times 60}{160} = 30$$

Rule: (c) *The area of a rectangle is equal to the product of two adjacent sides.* This has already been learned in denominate numbers. The fact that all land measurements are made in the rectangular system makes the rectangle an extremely important figure.

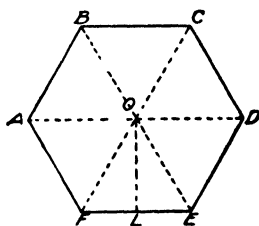


Fig. 34 Hexagon.

Areas of Other Polygons. In finding the areas of polygons, other than the triangle and quadrilateral, the system of dividing the figure into triangles is used. For example, the regular *hexagon*, Fig. 34, consists of six equal equilateral triangles, and consequently the area of the hexagon will be $\frac{6(FE \times OL)}{2} = 3(FE \times OL)$.

Rule: (d) *The area of a regular hexagon is equal to three times the product of one side (which is the same as $\frac{1}{2}$ the perimeter) by the perpendicular distance from the side to the center of the hexagon.*

The perpendicular distance from the center to one side of a regular polygon is called the *apothem*.

Example. The perimeter of a hexagon is 36 feet. Find the perpendicular distance and the area of the hexagon.

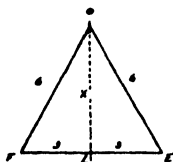
Solution. If the perimeter of the hexagon is given as 36 feet, one side (as $A B$) is $\frac{1}{6}$ of 36, or 6 feet.

Since the triangles are equilateral, the distance from any point (A, B, C, D, E, F) to the center is equal to the length of one side, or 6 feet.

To find the perpendicular distance from the center to one side, consider one of the triangles, as $F O E$.

Note how the altitude, $O L$, divides the triangle into two equal right-angled triangles, and the base, $F E$, into two equal parts.

Find $O L$ by the law of the right triangle, page 115. (See Example 2, page 116.)



$$\begin{aligned} 6^2 - 3^2 &= x^2 \\ 36 - 9 &= x^2 \\ x^2 &= 27 \\ x &= \sqrt{27} = 5.196 + \end{aligned}$$

The perpendicular distance is 5.196+ feet.

The area of the hexagon is $3 \times 6 \times 5.196$, or 93.528 sq. ft.

Answers: 5.196 ft.; 93.528 sq. ft.

PROBLEMS FOR PRACTICE

1. What is the area of the end of a regular hexagonal drain tile whose sides are 2 inches in length and whose perpendicular distance from the center to a side is 1.7 inches? Ans. 10.2 sq. in.

2. How much would it cost to lay a concrete sidewalk 8 feet wide and 525 feet long at 20 cents per square foot? Ans. \$840

3. How many square yards of plastering are needed for walls and ceiling of a room 10×12 feet and 9 feet high; the room contains a door $3\frac{1}{2} \times 7$ feet and two windows $3\frac{1}{2} \times 6$ feet? Ans. 50-sq. yd.

4. An irregular-shaped room has two parallel sides, one 14 ft. 8 in. long, the other 20 ft. 3 in. long. One end of the room is perpendicular to the two sides and is 12 ft. 5 in. long. What will it cost to paint the floor at 25 cents per square yard? Ans. \$6.02+

5. A trough 2 feet deep with equally sloping sides is constructed. One end is 5 feet across the top and 4 feet across the bottom. How many square feet of tin will be required for the end? Ans. 9 sq. ft.

CIRCLES

82. A *circle* is a plane figure bounded by a curved line called the *circumference*, every point of which is equally distant from a point within called the *center*, Fig. 35.

A *diameter* of a circle is a straight line drawn through the center, terminating at both ends in the circumference.

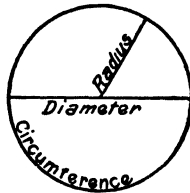


Fig. 35. Circle.

A *radius* of a circle is a straight line joining the center and the circumference. All radii of the same circle are equal and their length is always one-half that of the diameter.

An *arc* is any part of the circumference of a circle.

An arc equal to one-half the circumference is called a *semi-circumference*.

A *chord* is a straight line joining the extremities of an arc, Fig. 36. When a number of chords form the sides of a polygon, the polygon is said to be *inscribed in the circle*.

A *segment* of a circle, Fig. 37, is the area included between an arc and a chord.

A *sector* is the area included between an arc and two radii drawn to the extremities of the arc, Fig. 37.

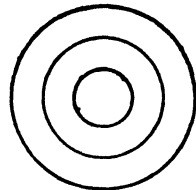
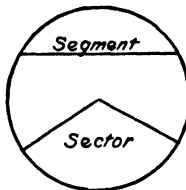
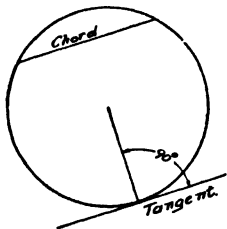


Fig. 36. Chord and Tangent. Fig. 37. Segment and Sector. Fig. 38. Concentric Circles.

A *tangent* is a straight line of unlimited length which touches the circumference at only one point, called the *point of tangency* or

point of contact, Fig. 36. It may be proved that a tangent is perpendicular to a radius drawn to the point of tangency.

Concentric circles are circles having the same center, Fig. 38.

Relation of the circumference to the diameter. If the lengths of the circumference and the diameter of a circle are carefully measured, the ratio, $\frac{\text{circumference}}{\text{diameter}}$, will be found to have a fixed value, whatever circle may be selected. This constant ratio has been given the name π (Greek letter *pi*) and has a value of 3.1416 (approximately $\frac{22}{7}$).

Therefore, *the circumference of any circle is equal to π times the diameter.*

The formula may then be written

$$C = \pi \times d$$

When the circumference is given, the diameter may be found by dividing by π .

$$d = \frac{C}{\pi}$$

The value $\frac{22}{7}$, or $3\frac{1}{7}$, may be used for rough calculations, but 3.1416 is more nearly correct, and should be used when greater accuracy is desired. Sometimes the value 3.14159 is given and used.

In many cases the value $\frac{22}{7}$ may be used with safety as the error is only about .04%. Then, to find the circumference when the diameter is known, *multiply the diameter by $\frac{22}{7}$* . Conversely, to find the diameter when the circumference is known, *divide the circumference by $\frac{22}{7}$* .

83. Areas of Circles. The circle may be divided into a number of equal sectors, Fig. 39, that are essentially triangles. The radius of the circle represents the altitude of these triangles, and the arc represents the base. Hence the area of all the sectors, Fig. 40 (which

is the area of the circle), is one-half the radius multiplied by the sum of all the arcs, which is the circumference. From this explanation the rule is obtained.

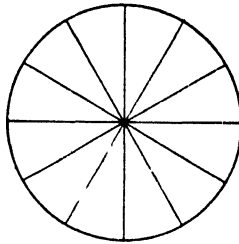


Fig. 39.

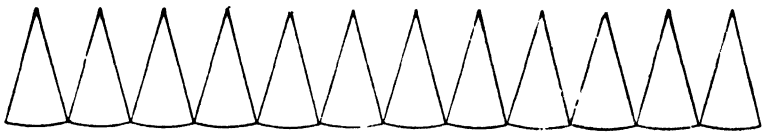


Fig. 40.

Rule: *The area of a circle is equal to the circumference multiplied by one-half the radius.*

$$\text{Area} = \frac{1}{2} (C \times r), \text{ or } C \times \frac{r}{2}$$

But you have already had the rule $C = \pi \times d$. Substituting this for C in the formula for area

$$\text{Area} = \pi \times d \times \frac{r}{2}$$

Since diameter

$$= 2r$$

$$\text{Area} = \pi \times 2r \times \frac{r}{2}$$

or

$$\text{Area} = \pi \times r \times r$$

This in turn gives the formula

$$A = \pi r^2$$

This is the most commonly used formula and for many the most easily remembered.

In terms of the diameter the formula becomes

$$\text{Area} = \pi \times d \times \frac{d}{4}$$

because

$$r = \frac{d}{2}, \quad \frac{r}{2} = \frac{d}{4}$$

From this formula the rule is obtained that the area equals .7854 times the diameter squared, because

$$\pi \times d \times \frac{d}{4} = \frac{1}{4} (\pi \times d \times d)$$

$$\frac{1}{4} \text{ of } 3.1416 = .7854 \text{ and } d \times d = d^2$$

It is well to remember this rule also. The formula is

$$A = \frac{1}{4} \pi d^2$$

From the formula $A = \pi r^2$, the radius equals the square root of the quotient when the area is divided by π . Or, in formula

$$r = \sqrt{A \div \pi}$$

Likewise from the formula $A = \frac{1}{4} \pi d^2$ the value of the diameter may be found.

$$d = \sqrt{A \div \frac{1}{4} \pi} = \sqrt{A \div .7854}$$

Examples. 1. Find the radius of a circle whose area is 392.70 square rods.

Ans. 11.18 rd.

$$\begin{aligned} r &= \sqrt{392.7 \div 3.1416} \\ &= \sqrt{125} = 11.18 + \end{aligned}$$

2. Find the diameter of a circle whose area is 78.54 square inches.

$$d = \sqrt{A \div \frac{1}{4} \pi} = \sqrt{78.54 \div .7854} = \sqrt{100} = 10$$

Ans. 10 in.

It has been shown that a line has but one dimension, and a surface or area, two. If this be true, then an expression representing a length contains *one* dimension and an expression representing area contains *two* dimensions. The circumference of a circle is πd , in which the diameter is the *dimension* and π is a ratio, or pure number. Again, the area of a circle is πr^2 , in which $r \times r$ gives the *two* dimensions; similarly the formula for area of a triangle is $\frac{1}{2}(b \times a)$, in which b and a are the dimensions.

EXERCISES

Find the circumference if the diameter is:

- | | |
|-----------|----------------|
| 1. 10 in. | 3. 5 ft. |
| 2. 12 ft. | 4. 2 ft. 6 in. |

Find the diameter when the circumference is:

- | | |
|-----------------|-------------------|
| 1. 126.7112 ft. | Ans. 40 ft. 4 in. |
| 2. 94.248 in. | Ans. 30 in. |
| 3. 87.9648 yd. | Ans. 28 yd. |
| 4. 39.27 ft. | Ans. 12 ft. 6 in. |

When the circumference equals 125.664 feet, what is the radius?

Ans. 20 ft.

Circumference = C ; diameter = d ; radius = r ; area = A

Find the area if

- | | |
|---------------|-------------------|
| 1. $d=10$ rd. | 5. $C=3.1416$ ft. |
| 2. $r=10$ rd. | 6. $d=25$ yd. |
| 3. $d=18$ ft. | 7. $C=94.248$ in. |
| 4. $r=60$ ft. | 8. $r=20$ yd. |

The student is recommended to use the value for π , 3.1416. When π is written together with a figure or a letter, the multiplication sign is understood between them. Thus, 2π means 2×3.1416 , and πr^2 means 3.1416 times the square of the radius.

PROBLEMS FOR PRACTICE

- When the diameter of a circle is $7\frac{3}{4}$ inches, what is the radius? Ans. $3\frac{7}{8}$ in.
- What is the circumference of a 4-foot wheel? Ans. 12.5664 ft.

3. How far will a 6-foot wheel go in making 110 revolutions?
Ans. 2073.456 ft.
4. How many revolutions will the same wheel make in going 1 mile?
Ans. 280.112+
5. The diameter of a pulley is 18 inches. What is the circumference?
Ans. 56.5488 in.
6. The bottom of a tank is circular and is 3 feet in diameter. What is the area of the bottom?
Ans. 7.0686 sq. ft.
7. What is the area of one side of a flat circular ring whose outside diameter is 8 inches and inside diameter 4 inches? Ans. 38- sq. in.
8. A carriage wheel 4 feet in diameter makes how many revolutions in traveling 1,000 feet?
Ans. 79.57+rev.
9. The circumference of a circular chimney is 47.12 feet. What is its diameter?
Ans. 14.99+ ft.
10. What will it cost to build a walk 5 feet wide around a circular flower bed 24 feet in diameter, at 35 cents per square yard?
Ans. \$17.71

SOLIDS

84. A *solid* has three dimensions—length, breadth, and thickness. The most common forms of solids are *prisms*, *cylinders*, *pyramids*, *cones*, and *spheres*.

PRISMS

85. A *prism* is a solid having two opposite faces, called *bases*, which are equal and parallel, and other faces, called *lateral faces*, which are parallelograms. The *altitude* of a prism is the perpendicular distance between the bases.

Prisms are called *triangular*, *rectangular*, *hexagonal*, etc., according to the shape of the bases. Further classifications are as follows:

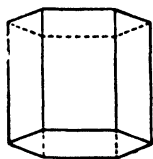


Fig. 41. Right Prism.

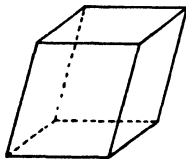


Fig. 42. Parallelepiped.

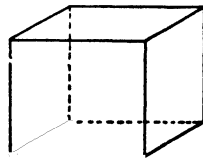


Fig 43. Rectangular Parallelepiped.

A *right* prism is one whose lateral faces are perpendicular to the bases, Fig. 41.

A *regular* prism is a right prism having regular polygons for bases.

A *parallelepiped* is a prism whose bases are parallelograms, Fig. 42. If all the edges are perpendicular to the bases, it is called a *right parallelepiped*.

A *rectangular parallelepiped* is a right parallelepiped whose bases and lateral faces are rectangles, Fig. 43. When these faces are all squares the prism is called a *cube*.

86. Areas of Prisms. The *area of the bases* of any given prism is to be found by the method already given for that particular shape of base. The *lateral area*, as each face is a parallelogram, will be the sum of the products of each base line by the altitude. This is equivalent to the product of the perimeter of the base by the altitude. The *total area* will evidently be the sum of the areas of the two bases and the lateral faces.

Example. Find the total area of a box 3 feet long, 2 feet wide, and 6 feet deep.

Solution. The area of one base is 2×3 or 6 sq. ft., and of two, 12 sq. ft.

The perimeter of the base $= 2 + 2 + 3 + 3 = 10$ ft.; the altitude $= 6$ ft.

The lateral area $= 10 \times 6 = 60$ sq. ft.

The total area $= 60 + 12 = 72$ sq. ft. Ans.

The total area of a cube is the sum of the six square surfaces that bound it. The line where any two of the surfaces meet is called the *edge* of the cube.

87. Volumes of Prisms. It has already been made clear in the discussion of *Cubic Measure* in denominate numbers that the measurement of volume involves the product of the three dimensions of the figure and is determined by the number of times the unit of cubic measure, such as the cubic inch, cubic centimeter, or cubic foot, is contained in the volume under consideration. If *A*, Fig. 44, represents a cubic foot, it will be noticed that *A* is contained four times in the bottom layer of *B*. As there are three layers, *A* is contained a total of twelve times in *B*, that is, the volume of *B* is 12 cubic feet.

But the area of the base of *B* is 4 sq. ft. and the altitude is 3 ft. The product of these two gives 12 cu. ft., the same as obtained by the use

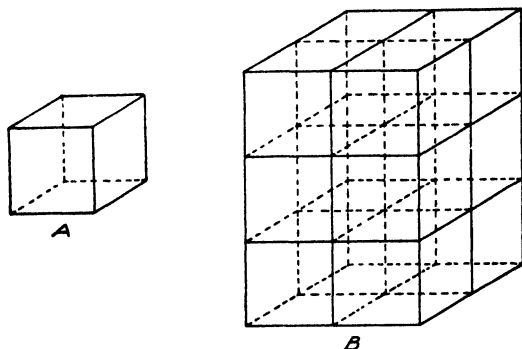


Fig. 44. Volume of a Rectangular Solid.

of the unit *A*. Rule: *The volume of any prism is equal to the product of the area of the base by the altitude.*

If *V* represents volume; *l*, length; *w*, width; and *h*, height, we have the formula

$$V = l \times w \times h$$

PROBLEMS FOR PRACTICE

1. What is the weight of a cast-iron block 6 inches long, 5 inches wide, and 3 inches high? The block is a rectangular parallelepiped and weighs .26 pounds per cubic inch. Ans. 23.4 lb.

2. How many square feet of sheet copper will be required to make an open rectangular tank 7 feet long, 3 feet wide, and $1\frac{1}{2}$ feet deep, allowing 10% extra for waste? Ans. 56.1 sq. ft.

3. How many cubic feet of concrete will be required to construct a column 15 feet high having a hexagonal cross section. The cross section is $1\frac{1}{2}$ feet long on each side of the hexagon and the perpendicular distance from center to each side is 1.3 feet. Ans. 87.75 cu. ft.

4. What will it cost to paint the above column, exclusive of the base and top, at 35 cents per square yard? Ans. \$5.25

5. Find the volume of a triangular prism whose altitude is 20 feet and each side of the base 4 feet. Ans. 138.4 cu. ft.

6. Find the total area of the prism given in Problem 5. Ans. 253.84 sq. ft.

CYLINDERS

88. A *cylinder* is a solid having as bases two equal parallel plane surfaces and as its lateral face the continuous surface generated by a straight line connecting the bases and moving along their cir-

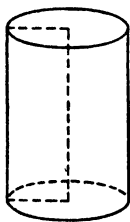


Fig. 45 Right Cylinder.

cumferences. The bases are usually circles and such a cylinder is called a *circular cylinder*.

A *right cylinder*, Fig. 45, is one whose side is perpendicular to the bases.

89. You may make a figure in the form of a cylinder by rolling a sheet of paper or a card until its opposite sides meet. Thus you see that the length of the card or paper is the altitude of the cylinder, and the width is the distance around, or the perimeter, of the cylinder. You can now see that the lateral area of the cylinder is equal to the product of the circumference of the base by the altitude.

$$\text{Lateral area} = C \times a, \text{ or } Ca$$

The total area is evidently equal to the sum of the areas of the bases and the lateral area.

Example. How much tin will be required to make a cylindrical can, 3 inches in diameter and 4 inches high, allowing nothing for seams and waste?

Solution. The can is a right cylinder the circumference of whose base is $\frac{22}{7} \times 3 = \frac{66}{7} = 9.43$ inches, nearly. Now, the radius of the base is $3 \div 2 = 1\frac{1}{2}$ inches, and the area of the base $\frac{22}{7} \times (1\frac{1}{2})^2 = 7.07$ sq. in.,

The lateral area is $9.43 \times 4 = 37.72$ sq. in.

The total area equals $37.72 + (2 \times 7.07) = 37.72 + 14.14 = 51.86$ square inches.

90. Volumes of Cylinders. *The volume of a cylinder is equal to the product of the area of one base by the altitude.*

$$V = a \times \pi r^2$$

Examples. 1. Find the volume of a cylinder the diameter of whose base is 20 inches, and whose altitude is 30 inches.

Solution. You have given the diameter of the base, 20 inches, and the altitude of the cylinder, 30 inches. First find the area of the base.

$$\text{Area} = \pi r^2$$

When diameter = 20 inches, $r = 10$

$$\pi r^2 = 3.1416 \times 100 = 314.16$$

$$V = 314.16 \times 30 = 9424.8 \text{ cu. in.}$$

$$9424.8 \div 1728 = 5.45 + \text{cu. ft., or } 5 \text{ cu. ft. } 784.8 \text{ cu. in.}$$

2. A piece of iron weighing 30 pounds has 4 holes 6 inches deep and 1 inch in diameter bored into it. What is the weight of the piece after the boring has been completed, iron weighing .26 pounds per cubic inch?

Solution. Each hole has a radius of $\frac{1}{2}$ inch and its volume is then $\frac{22}{7} \times (\frac{1}{2})^2 \times 6 = 4.7$ cu. in.

There are 4 holes making $4 \times 4.7 = 18.8$ cu. in.

The weight of the material cut out is $18.8 \times .26 = 4.9$ lb.

Then the final weight is $30 - 4.9 \text{ lb.} = 25.1 \text{ lb.}$

PROBLEMS FOR PRACTICE

1. What is the capacity of a cylindrical tank 9 feet long and 5 feet in diameter, inside measurements?

2. If a gallon contains 231 cubic inches, what must be the diameter of a cylindrical tank 10 feet high which will hold 1,000 gallons?
Ans. 4.1 + ft.

3. Find the number of square feet of material necessary for a straight piece of copper pipe 10 feet long and 12 inches in diameter.

PYRAMIDS

91. A *pyramid* is a solid whose base is a polygon and whose sides are triangles.

The vertices of the triangles meet to form the *vertex* of the pyramid.

The *altitude* of the pyramid is the perpendicular distance from the vertex to the base.

Pyramids are named according to the kind of polygon forming the base, namely, triangular, quadrangular, Fig. 46; pentagonal, Fig. 47; hexagonal, Fig. 48.

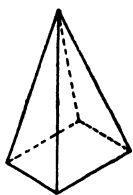


Fig. 46 Pyramid

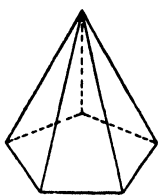


Fig. 47 Regular Pyramid

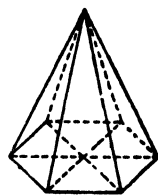


Fig. 48 Hexagonal Pyramid

A *regular pyramid* is one whose base is a regular polygon and whose vertex lies in a perpendicular erected at the center of the base, Fig. 47, Fig. 48, Fig. 49.

The *slant height* of a regular pyramid is a line drawn from the vertex perpendicular to a side of the base. (See the line OF , Fig.

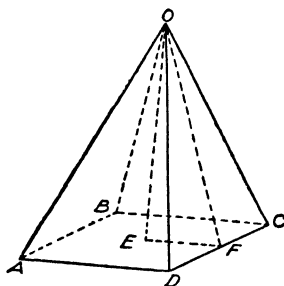


Fig. 49 Pyramid Showing Altitude and Slant Height.

49.) In other words, it is the altitude of one of the triangles which form the sides.

The *lateral edges* of a pyramid are the intersections of the triangular sides, called the *faces*.

92. Areas of Pyramids. (a) The *lateral area* is the combined area of all the triangles forming the sides.

The area of each triangle is equal to the product of the base by one-half the altitude. Therefore the lateral area of a pyramid is found by adding the products of each side of the base by one-half the same altitude; that is, it is equal to the perimeter of the base multiplied by one-half the slant height. If the slant height is not given it can usually be found by means of the law of the right triangle.

(b) The *total area* of a pyramid is equal to the sum of the lateral area and the area of the base.

Example. Find the lateral area and total area of the pyramid shown in Fig. 49, if it has an altitude $O E$ of 12 feet and each side of the square base is 8 feet.

Solution. $E F = 4$ feet. Since the angle $O E F$ is a right angle

$$\begin{aligned}\overline{OF}^2 &= \overline{OE}^2 + \overline{EF}^2 \\ &= 144 + 16 \\ &= 160 \\ OF &= \sqrt{160} = 12.65 \text{ (nearly)}\end{aligned}$$

$$\begin{aligned}\text{The lateral area} &= \frac{12.65 \times 32}{2} \\ &= 202.4 \text{ square feet. Ans.}\end{aligned}$$

The area of the base is $8 \times 8 = 64$ sq. ft., making the total area $64 + 202.4 = 266.4$ sq. ft. Ans.

93. Volumes of Pyramids. It may be shown that any triangular prism may be divided into three equivalent triangular pyramids, having bases and altitudes equal to those of the prism; and therefore the volume of each pyramid must be one-third of the volume of the prism. But the volume of a prism is equal to the product of the area of the base by the altitude; therefore, *the volume of a pyramid is equal to the product of the area of the base by one-third of the altitude.*

$$V = \text{Area of Base} \times \frac{1}{3} \text{ Altitude}$$

or
$$V = \frac{B \times h}{3}$$

Here B = Area of base and h = height or altitude of the pyramid.

Example. What is the volume of a triangular pyramid whose base is 3 feet on a side, and whose altitude is 9 feet?

Solution. The base is an equilateral triangle. Find the altitude.

$$3^2 - \left(\frac{3}{2}\right)^2 = 9 - \frac{9}{4} = \frac{27}{4}$$

$$\sqrt{\frac{27}{4}} = \frac{5.196}{2} = 2.598 +$$

$$\text{Area of base} = \frac{3 \times 2.598}{2} = 3.897 +$$

$$\text{Volume} = 3.897 \times \frac{9}{3} = 11.691 + \text{cu. ft.} \quad \text{Ans.}$$

PROBLEMS FOR PRACTICE

1. A triangular pyramid is 9 inches in height and each side of the base is 4 inches long. Find the volume. Ans. 20.76 cu. in.
2. What is the volume of a square pyramid one side of whose base is 4 inches and whose height is 4 inches? Ans. $21\frac{1}{3}$ cu. in.
3. Find the total area of a square pyramid whose slant height is 28 inches and one side of whose base is 8 inches. Ans. 512 sq. in.
4. A regular hexagonal pyramid has an altitude of 18 inches, and each side of the hexagon is 6 inches long. Find total area and volume of pyramid. Answers: 430.84 + sq. in.; 561.16 + cu. in.

CONES

94. A *cone* is a solid bounded by a conical surface and a plane which cuts the conical surface. It may be considered as a pyramid with an infinite number of sides.

The conical surface is called the *lateral area*, and the plane is called the *base*. The conical surface tapers to a point called the *vertex*.

The *altitude* of a cone is the perpendicular distance from the vertex to the base.

An *element of the cone* is any straight line from the vertex to the perimeter of the base. Such a line is the *slant height* of the cone.

A *circular cone* is a cone whose base is a circle, Fig. 50.

A *right circular cone*, or *cone of revolution*, Fig. 51, is a cone whose axis is perpendicular to the base. It may be generated by the revolution of a right triangle about one of the sides as an axis.

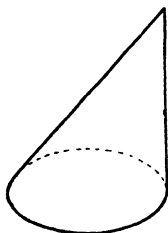


Fig. 50. Circular Cone.

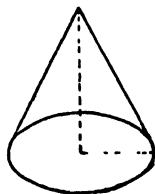


Fig. 51. Cone of Revolution.

Cut a right triangle of cardboard. Stand it upright on its base, and turn it once around, keeping the altitude in the same position. You will make a circle whose radius is the base of the right triangle.

The circle is the *base* of a cone of revolution, the hypotenuse of the triangle is the slant height of the cone, and the altitude of the triangle is the altitude of the cone. See Fig. 51.

95. Areas of Cones. The *lateral area* of a cone is found in the same way as in the case of a pyramid. *Multiply the perimeter of the base by one-half the slant height.*

Example. The base of a cone is 12 inches in diameter and the slant height is 16 inches. What is the total area?

$$\begin{aligned}
 \text{Solution. Perimeter of base} &= 12 \times \frac{22}{7} \\
 &= 37.71 \text{ inches} \\
 \text{Lateral area} &= 37.71 \times \frac{16}{2} \\
 &= 301.68 \text{ square inches}
 \end{aligned}$$

To find the *total area*, add to this the area of the base. In the above example, the area of the base is

$$\frac{22}{7} \times 36 = 113.14 \text{ sq. in.}$$

$$\text{Total area} = 113.14 + 301.68 = 414.82 \text{ square inches.}$$

96. Volumes of Cones. As a cone may be considered a pyramid with an infinite number of sides, it follows from Section 93 that *the volume of a cone is equal to the product of the area of the base by one-third the altitude.*

Example. The altitude of a cone is 18 inches and the radius of the base is 2 inches. What is the volume?

$$\text{Area of base} = \frac{22}{7} \times 4 = 12.57$$

$$\text{Volume} = 12.57 \times 6 = 75.42 \text{ cubic inches}$$

In case the altitude is not known, it may be found if the slant height and the radius of the base are known.

PROBLEMS FOR PRACTICE

1. Find the volume of a cone whose altitude is 12 inches and the diameter of whose base is 6 inches. Ans. $113\frac{1}{2}$ cu. in.
2. Find the total area of a cone having an altitude of 21 feet and a base whose diameter is 15 feet. Ans. 702.4 sq. ft.
3. Find the lateral area of a cone whose diameter is 17 feet 6 inches, and the slant height 30 feet. Ans. 824.67 sq. ft.
4. Find the solid contents of a cone whose altitude is 24 feet, and the diameter of whose base is 30 inches. Ans. $39.28 +$ cu. ft.

SPHERES

97. A *sphere* is a solid bounded by a curved surface, every point of which is equally distant from a point within called the *center*.

The *diameter* is a straight line drawn through the center and having its extremities in the curved surface. The *radius* is the straight line from the center to a point on the surface; it is equal to one-half the diameter.

A *plane is tangent to a sphere* when it touches the sphere in only one point. A plane perpendicular to a radius at its outer extremity is tangent to the sphere.

98. Areas of Spheres. To find the *area* of the surface of a sphere, *multiply the square of the diameter by $\frac{22}{7}$.*

Example. Find the area of the surface of a sphere which is 9 inches in diameter.

$$9^2 = 81. \quad 81 \times \frac{22}{7} = 254.6 \text{ square inches}$$

To find the *diameter* when the surface is given, *divide the surface by $\frac{22}{7}$ and extract the square root of the quotient.*

Example. What is the diameter of a sphere whose area is 113.14 square feet?

$$113.14 \div \frac{22}{7} = 36 \text{ nearly. } \sqrt{36} = 6 \text{ feet}$$

99. Volumes of Spheres. To find the *volume* of a sphere, multiply the cube of the diameter by $\frac{\pi}{6}$ (equal to, $\frac{11}{21}$ nearly).

Examples. 1. A sphere is 10 inches in diameter; what is the volume?

$$10^3 = 1000. \quad 1000 \times \frac{11}{21} = 523.8 \text{ cubic inches}$$

2. Find the volume of a sphere having a diameter of 7 feet.

$$7^3 = 343; \quad 343 \times \frac{11}{21} = 179\frac{2}{3} \text{ cubic feet}$$

PROBLEMS FOR PRACTICE

1. The surface of a sphere contains 314.16 square inches. What is the diameter? Ans. 10 in.

2. What will be the volume of an aluminum sphere which has a diameter of 15 inches? Ans. 1.02 cu. ft.

3. How much will a sphere of cast iron weigh if it is 3 feet in diameter and iron weighs 450 pounds per cubic foot? Ans. 6364 $\frac{2}{3}$ lb.

4. How much will it cost to paint the sphere in Problem 3 at 25 cents per square yard? Ans. 78 $\frac{1}{4}$ ¢.

5. How much less surface has a sphere which is 12 inches in diameter than a cube with an edge of 2 feet? Ans. 20.85+sq. ft.

6. A spherical balloon, when inflated, has a diameter of 50 feet. Find the surface and the number of cubic feet of gas contained in the bag. Answers: 7857 $\frac{1}{2}$ sq. ft.; 65,476 $\frac{4}{11}$ cu. ft.

GRAPHS

The method of presenting the results of research, as well as the compiling of statistics and various tables of information, by pictorial means is familiar to all readers of newspapers and magazines. Some of the most common means of such presentation are the line, or bar, the sector, and the curve.

The following graph, Fig. 52, represents the yield of corn in several states: Kansas, 174,225,000; Nebraska, 182,616,000; Indiana, 174,600,000; Iowa, 432,021,000; Illinois, 426,320,000.

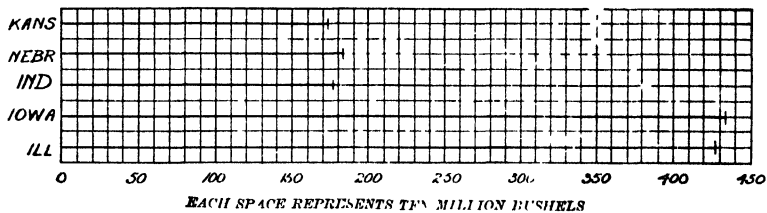


Fig. 52 Graph Showing Yield of Corn in Five States.

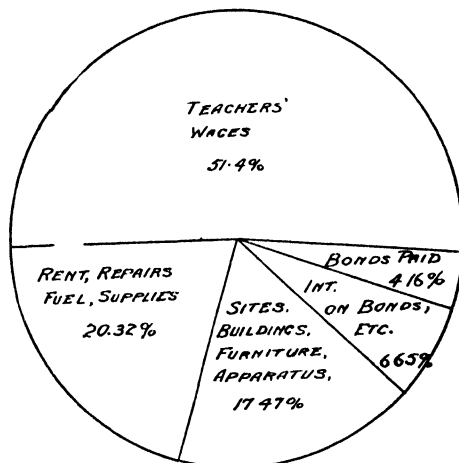


Fig. 53. Graph Showing Distribution of School Funds.

The graph represented in Fig. 53 shows the distribution of \$13,649,000 spent for school purposes in a western state.

By means of squared paper the results of experiments and observations of various kinds can be represented by lines and curves.

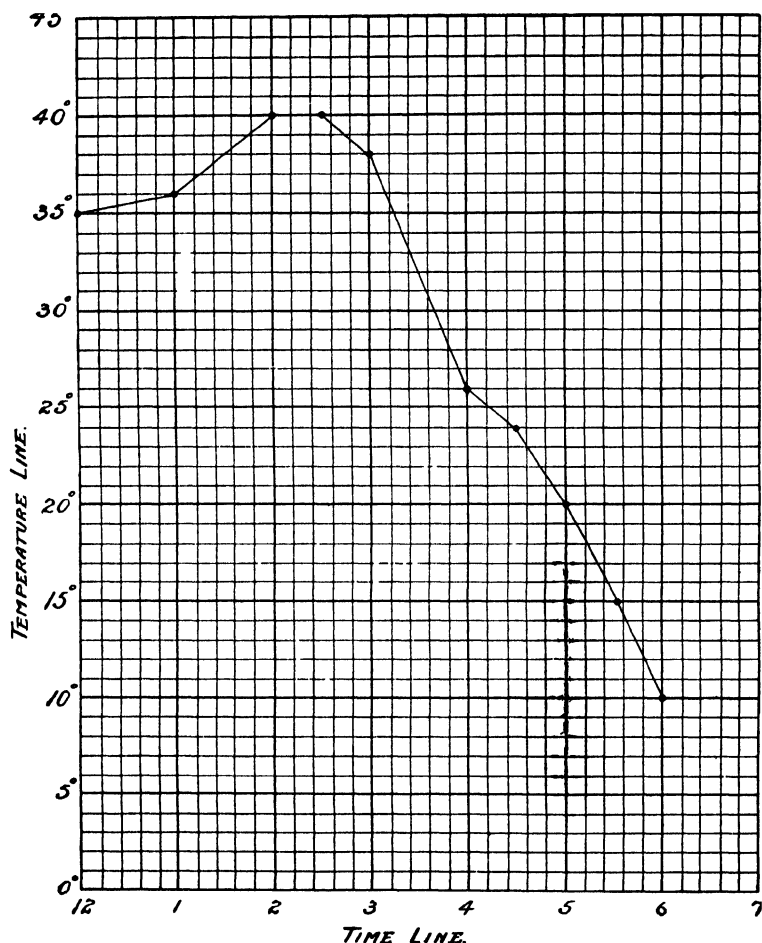


Fig 54 Curve Showing Temperature for Six Hours.

A curve showing the temperature for six hours, from noon to 6 P. M., is shown, Fig. 54. Notice that the hours are shown on a horizontal line, or axis, and the temperature on a vertical line, or axis.

Time	Temperature
12:00 noon	35°
1:00	36°
2:00	40°
2:30	40°
3:00	38°
4:00	26°
4:30	24°
5:00	20°
5:30	15°
6:00	10°

Curves can be drawn showing the cost of doing contract work or of operating a certain machine; the variation in temperature of a fever patient in a hospital; the fluctuations in the price of wheat; or the cost of living. The range of application is unlimited. The student can readily see that curves are valuable records for commercial and professional men, as well as for trained engineers, skilled machinists, and statisticians.

ILLUSTRATIVE CURVES

Curves for Squares and Square Roots. In Fig. 55 is shown a curve expressing the relation of numbers and the squares of those numbers. Curves of this kind are very practical, and, as can be readily seen, may be used to save time in all kinds of mathematical work. Compare, for example, the time required to find the *square* of 9.8 mathematically and the time to take the result directly from the above curve. To understand thoroughly the method of using such a curve, the following explanation will be helpful. Five divisions on the horizontal axis equal one unit and, therefore, each division equals $\frac{1}{5}$, or $\frac{2}{10}$, or .2. On the vertical axis there are five divisions to ten numbers and, therefore, each division equals $\frac{1}{5}$ of 10, or 2. First find on the horizontal axis the point whose value is 9.8. It will be between 9 and 10, just one line short of division 10. Follow this line upward until it meets the curve and from this point go horizontally to the left until the vertical axis is reached. On this line read the value. It is between 90 and 100, three divisions above

90. Each division represents 2, therefore three divisions equal 6. This must be added to 90, giving as a final result 96 approximately.

To find the *square root* of a number the above method would be reversed, using the same curve. Find the number on the vertical

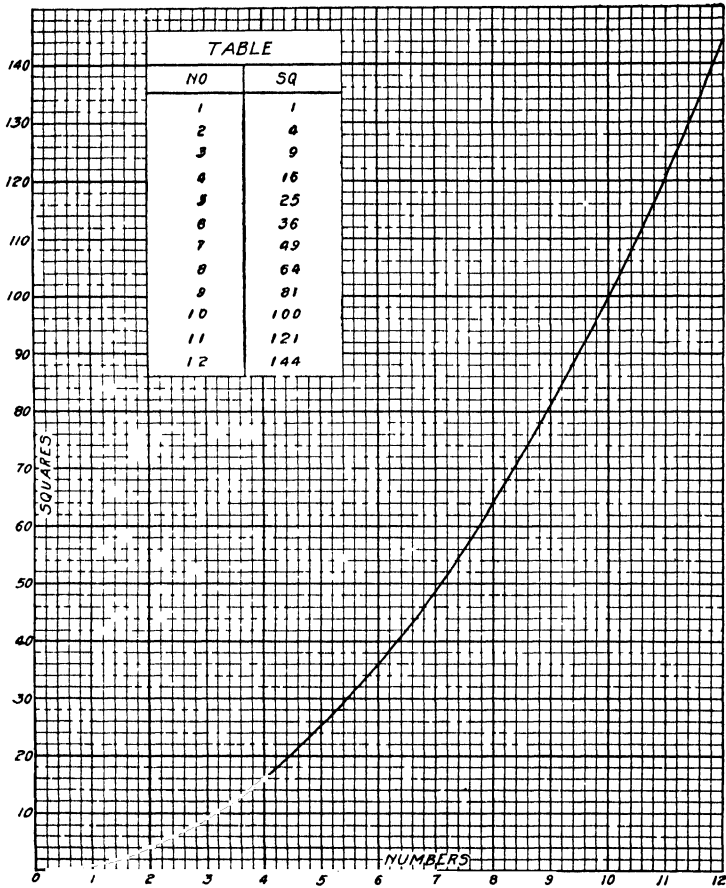


FIG 55 Curve of Squares and Square Roots

axis whose root is to be extracted, then follow horizontally across the sheet to the point of intersection with the curve and then downward to the horizontal axis. The value found there will be the square root; for instance, the square root of 96 = 9.8 approximately.

CURVE PROBLEMS FOR PRACTICE

1. Plot the curve from the following data for the cost of high-speed compound engines, using thousands of dollars for vertical values and horsepower for horizontal values. What will be the cost of a 750-horsepower engine of this type?

Dollars	Horsepower
6400	400
9600	600
12800	800
16000	1000
24000	1500
32000	2000
40000	2500
48000	3000

2. Plot the curve from the following data for the increase of Chicago's population, using population in thousands as vertical values and years for horizontal values. Assuming the same increase in the next decade as in the last, find the population of Chicago in 1920.

Population	Years
109,206	1860
178,492	1865
306,605	1870
400,500	1875
503,185	1880
665,000	1885
1,099,850	1890
1,366,813	1895
1,698,575	1900
1,949,116	1905
2,195,551	1910

3. Plot the curve from the following data for the areas and diameters of circles, using areas for vertical values and diameters for horizontal values. What is the diameter of a circle having an area of 40 square inches?

Area Square Inches	Diameter Inches
.7854	1
3.1416	2
7.0686	3
12.566	4
19.635	5
28.274	6
38.485	7
50.265	8
63.617	9
78.540	10
95.033	11
113.097	12

4. The average monthly rainfall in inches at a certain station in Nebraska for 35 years ending 1909 was as follows:

Jan. .42	May 2.86	Sept. 1.47
Feb. .49	June 3.41	Oct. 1.10
Mar. .81	July 2.90	Nov. .46
April 2.04	Aug. 2.37	Dec. .49

Represent by means of squared paper.

EXERCISES

Look at these expressions and see how many of them you can define *accurately*, in mathematical language: diameter, acute, equation, ratio, obtuse, arc, radius, sector, altitude, circumference, cube root, hypotenuse, perimeter, slant height, right angle, plane, solid, dimension, area, volume, trapezoid, lateral area, prism, pyramid, cylinder, cone, sphere.

Write the formulas for the following areas: triangle, square, trapezoid, circle, hexagon, right triangle, parallelogram, lateral area of a cone.

Write the formulas for the following volumes: cube, rectangular prism, cylinder, pyramid, cone.

Write these exercises and then check your own answers by referring to the book.

GENERAL SUGGESTIONS FOR THE SOLUTION OF PROBLEMS

In considering a problem there are several points that are especially important:

1. Read the problem carefully, several times, to make sure that you understand the exact meaning of it.

2. Analyze the problem, separating the information given from the result that is required, and determining how the result is to be obtained from the conditions given in the problem.

It is not always necessary to set down a complete analysis, but enough should be shown to indicate that the student has *mentally* gone through all the steps necessary for the solution. A complete *statement* should be set down. The equation method is recommended. Do not write out an analysis in words.

3. The solution should involve no unnecessary work. Cancellation and other short methods should be used if possible.

4. All arithmetical work should be carefully checked. The student must realize that *accuracy* is of the highest importance, and that to insure accuracy his work must always be checked. Arithmetical work that has an error in it is of no practical value. The check also gives the student a means of knowing for himself whether his work is right or not. In the real business of life he is not furnished with answers to his problems.

Another way to check on your work is to examine your result from the standpoint of *reasonableness*. By this is meant to read the problem carefully and then think whether your result seems to you a reasonable result for the conditions given in the problem. Often an error is detected at once by applying this test. It may be a mistake in calculation, or in the placing of a decimal point. Such mistakes are found by checking, and by applying the rule for placing the decimal point. If your result still seems wrong, go over your methods and see if you have used the right one in each step of the work. Verify each rule you have used.

5. It is often a good plan to state a problem in one step so as to shorten the work by cancelling.

Examples. 1. A canning factory can put up 300 cans of peas per hour. If an acre produces 640 cans of peas, how long will it

take the factory to put up the product of 75 acres, if the men work 8 hours a day, 6 days in the week? Give the answer in weeks.

$$\begin{array}{r}
 10 \\
 20 \\
 160 \\
 \hline
 75 \times 640 = 10 \\
 300 \times 8 \times 6 = 3
 \end{array}
 \quad \text{or } 3\frac{1}{3} \quad \text{Ans. } 3\frac{1}{3} \text{ wk.}$$

2. What is the expense of covering a floor with plain matting 1 yard wide, the room being 21 by $23\frac{1}{2}$ feet, the matting worth 40 cents a yard, allowing 6 inches extra on each strip?

$$\begin{array}{r}
 7 \quad 8 \\
 21 \times 24 \times .40 \\
 \hline
 9 \quad 3 \\
 22.40
 \end{array}$$

or $7 \times 8 \times .40 = 22.40$

A room 21 feet wide requires exactly 7 strips of matting.

$$23\frac{1}{2} \text{ ft.} + 6 \text{ in.} = 24 \text{ ft., or } 8 \text{ yd.}$$

Hence $7 \times 8 \times .40 = \text{cost of the matting.}$

The two steps of reasoning may be taken mentally and not set down.

In working with problems involving decimals, carry to as many places as the given quantities or the formulas show. For example, any problem in which you use π should be accurate to at least four decimal places.

"Accurate to two decimal places" refers to the *final* result, and not to the intermediate steps of a problem.

NOTE: For ready reference, common tables used in solving problems will be found on page 152 preceding the examination.

MISCELLANEOUS PROBLEMS

NOTE: These problems are *not* to be sent to the School. They are given so that the student may test his knowledge and progress.

1. Simplify:

$$(a) \frac{6\frac{3}{4} + (5\frac{1}{2} \times 3\frac{1}{2}) - 7\frac{1}{2}}{6\frac{2}{3} + 5 - 8\frac{1}{5}}$$

$$(b) (.26 \times .3 \times .02) \div (.5 \times .01 \times 1.04)$$

$$(c) (21\frac{1}{8} + 1\frac{5}{6}) \div (8 - 5\frac{2}{3}) \quad \text{Answers: (a) } 5\frac{2}{3}; (b) .3; (c) 1\frac{1}{2}$$

2. If evaporation from a tank is estimated at $6\frac{1}{4}\%$ in a week, how many gallons will be lost from a tank that has a capacity of 6400 gallons but which is only 75% full at the beginning?

Ans. 300 gal.

3. Find the number of hours it will take a locomotive running at the rate of 36 miles per hour to make the distance passed over by another locomotive running 9 hours at the rate of 48 miles per hour.

Ans. 12 hr.

4. A steel rail 30 feet long weighs 720 pounds. What is its weight per yard of length?

5. What is the area of a square one side of which is $9\frac{1}{2}$ feet?

Ans. $90\frac{1}{4}$ sq. ft.

6. What is the side of a square whose area is 64 square inches?

7. How many square feet of floor area in a freight car whose inside dimensions are 33 feet 6 inches by 8 feet 3 inches?

Ans. $276\frac{3}{8}$ sq. ft.

8. How many square feet of glass in a window containing 4 panes, each 18 inches by 36 inches?

Ans. 18 sq. ft.

9. How many square feet in the walls of a room 30 feet long, 18 feet wide and 16 feet high?

Ans. 1536 sq. ft.

10. What is the area of the ceiling of the same room?

Ans. 540 sq. ft.

11. Which is larger, a triangle having a base of $6\frac{1}{2}$ inches and an altitude of $3\frac{1}{4}$ inches, or a parallelogram whose base is $4\frac{1}{4}$ inches and whose altitude is $2\frac{3}{8}$ inches? How much?

Ans. The triangle. $\frac{1}{8}$ sq. in.

12. Find the area of a trapezoid whose altitude is 8 inches and whose parallel sides are respectively 18 inches and 24 inches.

Ans. 168 sq. in.

13. Find the area of a triangle whose base is 26 inches and whose altitude is 15 inches. Ans. 195 sq. in.

14. Find the base of a triangle whose altitude is 18 inches and whose area is 90 square inches. Ans. 10 in.

15. If the area of an obtuse triangle is 34 square inches and the altitude is $4\frac{1}{4}$ inches, what is its base? Ans. 16 in.

16. Find the area of a right triangle whose two legs are 5 inches and 12 inches and whose hypotenuse is 13 inches.

In finding the area of this triangle, which of the dimensions is unnecessary? Why?

17. How many pieces of zinc 4 inches $\times$ 6 inches can be cut from a sheet of zinc 3 feet $\times$ 6 inches? Ans. 9 pieces

(The sign $\times$ used as above means "by.")

18. If a sheet of copper 30 inches $\times$ 60 inches weighs 25 pounds, what is the weight per square foot? Ans. 2 lb.

19. What is the area of a parallelogram whose base is 14 feet and whose altitude is 8 feet? Ans. 112 sq. ft.

20. What is the altitude of a parallelogram whose base is 16 feet and whose area is 80 square feet? Ans. 5 ft.

21. Find the cost at 65 cents per square yard of a concrete platform rectangular in shape, 100 feet long and 25 feet wide.

Ans. \$180 $\frac{5}{8}$

22. The sides of a right triangle are 16 and 30 inches, respectively. What is the hypotenuse? Ans. 34 in.

23. A guy wire is fastened to the top of a wireless tower 454 feet high and is anchored to the ground 724 feet from the foot of the tower. Find the length of the wire. Ans. 854.57+ft.

24. One side of a right triangle is 9 feet, and the hypotenuse is 41 feet. Find the length of the other side. Ans. 40 ft.

25. A tree was broken off 10 feet from the ground and the top struck the ground 40 feet from the foot of the tree. What was the height of the tree? Ans. 51.23+ft.

26. If the circumference is $3\frac{1}{2}$ times the diameter, and the drive wheel of a locomotive is 6 feet in diameter, how many revolutions will it make in going 140 miles? Ans. 39,200 rev.

27. Soldiers marching quickstep take 120 steps per minute, averaging $2\frac{1}{2}$ feet each step. At this rate, how far will a company of soldiers march in an hour? Ans. $3\frac{2}{3}$ mi.

28. The diameters of two concentric circles are 20 feet and 30 feet. Find the area of the ring. Ans. 392.7 sq. ft.

29. Find the circumference and the area of the largest circle that can be drawn in an 8-foot square.

Answers: 25.133— ft.; 50.266— sq. ft.

30. The diagonal of a square field is 40 rods. How many acres are in the field? Ans. 5 A.

31. Find the cost, at 25 cents a rod, of building a fence around a 10-acre square field. Ans. \$40

32. How many cubic feet in a Portland cement pillar in the form of a right cylinder whose diameter is 12 inches and whose height is 15 feet? Ans. 11.781 cu. ft.

33. If Portland cement weighs 10 pounds per cubic foot, what is the weight of the pillar? Ans. 117.81 lb.

34. A cylindrical water tank 24 feet high has a capacity of 32,000 cubic feet. What is the diameter? Ans. 41.2 ft.

35. How many cubic yards must be removed in digging a tunnel 468 feet long, 21 feet wide, and 18 feet 6 inches deep?

Ans. 6734 cu. yd.

36. The volume of a rectangular parallelepiped is 100 cubic inches. The area of one end is 20 square inches. Find the length.

Ans. 5 in.

37. How many cubic feet of air are in a room 12 feet 6 inches long, 10 feet 8 inches wide, and 9 feet high? Ans. 1200 cu. ft.

38. Find the weight of a rectangular block of stone at 135 pounds per cubic foot, if the length is 10 feet and the other two dimensions are 2 feet and 5 feet.

39. How many tons of coal will a bin 10 feet by $6\frac{1}{2}$ feet by $7\frac{3}{4}$ feet hold, if one ton occupied 36 cubic feet?

Ans. 13 T. 16 cwt. $85\frac{5}{7}$ lb.

If the student can solve these problems with ease, he is ready to write the examination. On the other hand, if the problems give him serious trouble, a careful review of the textbook is needed. With due care in the preparation of the work given for practice, every student will be able to do excellent work in his examination paper.

LINEAR MEASURE

12 inches (in. or ")	= 1 foot (ft. or ')
3 feet	= 1 yard (yd.)
$5\frac{1}{2}$ yards or $16\frac{1}{2}$ ft.	= 1 rod (rd.), pole, or perch
320 rods or 5280 ft.	= 1 mile (mi.)

A *knot*, used in navigation, is equal to 1.15 miles.

A *fathom* is equal to six feet.

SQUARE MEASURE

144 square inches (sq. in.)	= 1 square foot (sq. ft.)
9 square feet	= 1 square yard (sq. yd.)
$30\frac{1}{4}$ square yards	= 1 square rod (sq. rd.)
160 square rods	= 1 acre (A.)
640 acres	= 1 square mile (sq. mi.)

CUBIC MEASURE

1728 cubic inches (cu. in.)	= 1 cubic foot (cu. ft.)
27 cubic feet	= 1 cubic yard (cu. yd.)
128 cubic feet	= 1 cord (cd.)
8 cord feet	= 1 cord (cd.)

LIQUID MEASURE

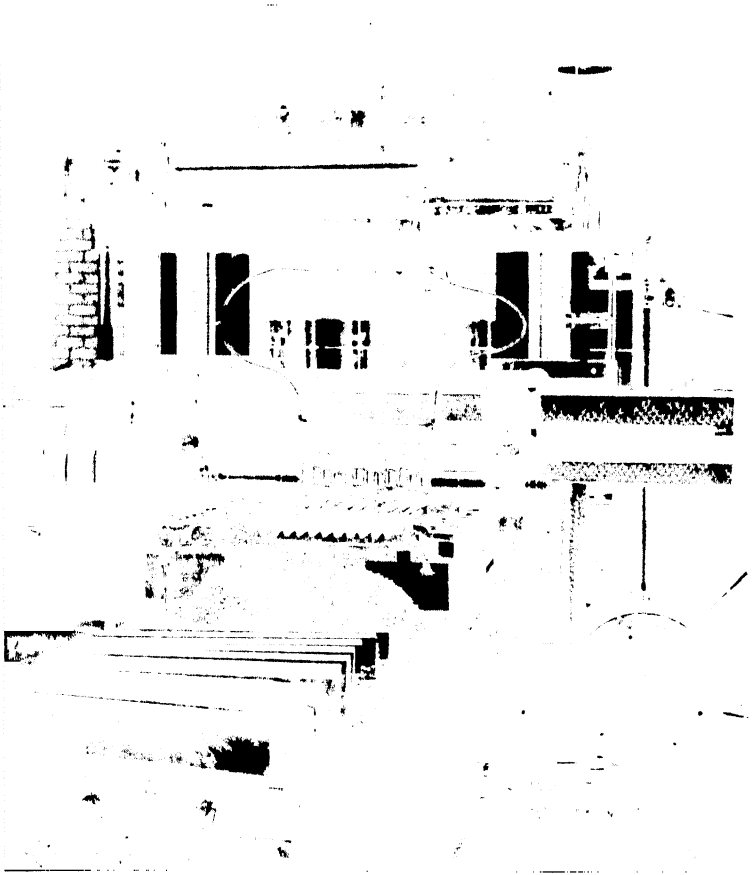
4 gills (gi.)	= 1 pint (pt.)
2 pints	= 1 quart (qt.)
4 quarts	= 1 gallon (gal.)
$31\frac{1}{2}$ gallons	= 1 barrel (bbl.)
2 barrels, or 63 gallons	= 1 hogshead (hhd.)

One U. S. standard gallon contains 231 cubic inches, and one gallon of water weighs approximately $8\frac{1}{3}$ pounds.

One cubic foot of water contains approximately 7.5 U. S. standard gallons and weighs 62.3 pounds.

SQUARES AND SQUARE ROOTS OF NUMBERS

Number	Square	Square Root	Number	Square	Square Root
1	1	1	51	2601	7.141
2	4	1.414	52	2704	7.211
3	9	1.732	53	2809	7.280
4	16	2	54	2916	7.348
5	25	2.236	55	3025	7.416
6	36	2.449	56	3136	7.483
7	49	2.646	57	3249	7.550
8	64	2.828	58	3364	7.616
9	81	3	59	3481	7.681
10	100	3.162	60	3600	7.746
11	121	3.317	61	3721	7.810
12	144	3.464	62	3844	7.874
13	169	3.606	63	3969	7.937
14	196	3.742	64	4096	8
15	225	3.873	65	4225	8.062
16	256	4	66	4356	8.124
17	289	4.123	67	4489	8.185
18	324	4.243	68	4624	8.246
19	361	4.359	69	4761	8.307
20	400	4.472	70	4900	8.367
21	441	4.583	71	5041	8.426
22	484	4.690	72	5184	8.485
23	529	4.796	73	5329	8.544
24	576	4.899	74	5476	8.602
25	625	5	75	5625	8.660
26	676	5.099	76	5776	8.718
27	729	5.196	77	5929	8.775
28	784	5.291	78	6084	8.832
29	841	5.385	79	6241	8.888
30	900	5.477	80	6400	8.944
31	961	5.568	81	6561	9
32	1024	5.657	82	6724	9.055
33	1089	5.745	83	6889	9.110
34	1156	5.831	84	7056	9.165
35	1225	5.916	85	7225	9.220
36	1296	6	86	7396	9.274
37	1369	6.083	87	7569	9.327
38	1444	6.164	88	7744	9.381
39	1521	6.245	89	7921	9.434
40	1600	6.325	90	8100	9.487
41	1681	6.403	91	8281	9.539
42	1764	6.481	92	8464	9.592
43	1849	6.557	93	8649	9.644
44	1936	6.633	94	8836	9.695
45	2025	6.708	95	9025	9.747
46	2116	6.782	96	9216	9.798
47	2209	6.856	97	9409	9.849
48	2304	6.928	98	9604	9.899
49	2401	7	99	9801	9.950
50	2500	7.071	100	10000	10



**HORIZONTAL INCLINED-RAIL MILLING MACHINE MILLING RACKS. STEEL
BARS SHOWN IN FOREGROUND PRIOR TO BEING
INSERTED IN MACHINE**

Courtesy of Ingersoll Milling Machine Company, Rockford, Illinois

LOGARITHMS

This is a day of efficiency and time-saving devices. Anyone who can devise some means for saving time on a job is well paid for his services. Competition is so great in business that time-saving devices are looked for continually. And as it is in business so it is in our study. We need every help we can get which will make our time count to the greatest advantage.

A shortcut method, called logarithms (pronounced log'a-rithms), has been devised which will not only save time in multiplying large numbers together, but it can also be used to divide, to extract the square root, the cube root, or any other root; to square a number, cube it, or figure out the exact amount for any other power of the given number; or it can be used for any combination of these operations. Take, for example, the following problem.

Find the value of $\sqrt{\frac{3678^4 \times .03257^2}{1679^3 \times 1.345^5}}$

The symbol in front of the problem is called the **square root** or **radical sign**. It means that the square root of the combined operations under the symbol is to be extracted. To find the square root of a given number means to find a number which when multiplied by itself will equal the given number. For instance, the square root of 9 is 3, for 3 times 3 equals 9.

The first number in the numerator is 3678. It has a small figure above it. This figure is called an **exponent** and means that the number is taken as many times as the exponent indicates. Each number in this problem has an exponent, and therefore each number is taken as a factor as many times as its exponent indicates.

To solve this problem we would have to multiply the first number in the numerator (3678) four times (that is, $3678 \times 3678 \times 3678 \times 3678$); then we would have to multiply the second number (.03257) two times (that is, $.03257 \times .03257$); and then we would have to multiply the results of these two operations together, which

would give us the combined numerator. Similarly, for the denominator we would have to multiply the first number in the denominator (1679) three times (that is, $1679 \times 1679 \times 1679$); we would have to multiply the second number in the denominator (1.345) five times; and then we would have to multiply these two results together to get the combined denominator. After we have done this, we would have to divide the combined numerator by the combined denominator. We would now have one number, which is the combination of all the terms under the **square root sign**. The final operation would be to extract the square root of this number.

You, no doubt, already appreciate how long a problem this would be to work out by arithmetic. When you have learned how to use the logarithmic tables, we will show you in detail just how to work a similar problem by logarithms and you will see how short it is. This short method also takes away the chance of many errors which are constantly being made in the operations of multiplying and dividing, therefore we feel sure that you will want to learn this method right away.

The invention of logarithms has been accorded to John Napier, who was a baron of Scotland. The tables, however, which we are using at the present time are called the Briggs Tables. Henry Briggs was a professor of geometry in London, England. He often visited Mr. Napier in order to get his ideas on this subject of logarithms. Then Briggs spent a large part of his time from 1615 to 1628 in compiling his tables.

The fact that it took one man a great many years to calculate and produce these tables is sufficient proof that it was a real job, and because a man was willing to spend years to produce a tool with which to make your work in Mathematics easier and shorter, you should be glad to use it. The important thing at this time, however, is for you to **learn how** to use this tool.

Read the following table carefully and note the exponents:

$10^1 = 10$ and the logarithm of 10 is 1

$10^2 = 100$ and the logarithm of 100 is 2

$10^3 = 1000$ and the logarithm of 1000 is 3

$10^6 = 1000000$ and the logarithm of 1000000 is 6

$10^x = N$ and the logarithm of N is x

In the last line x is the index of the power to which 10 must be raised in order to equal N , or x is the logarithm of N to the base 10.

Therefore, the common logarithm of a number is the exponent to which 10 must be raised to give the number.

The Logarithmic Tables give a systematic list of exponents of 10 with their corresponding numbers.

If a decimal exponent is used, then the logarithm has a decimal part, which is written in decimal form.

Thus $10^{1.6} = 39.81+$ and the logarithm is 1.6.

Thus, it is seen that when the number is an exact power of 10, the logarithm has no decimal part; and when the number is not an exact power of 10, the logarithm has a decimal part. These decimal parts are the parts which required such a long time for Mr. Briggs to calculate. You can multiply a number two or three times quite easily, but you cannot multiply it $2\frac{1}{2}$ or $2\frac{1}{3}$ times.

There are two parts to every logarithm. The first part, which is to the left of the decimal point, as figure 2 in 2.342817, is called the **characteristic**. The second part, which is to the right of the decimal point, as figures .342817, is called the **mantissa**. These are large words but you will soon learn to handle them easily. Characteristic is pronounced char'ac-ter'is-tic. Mantissa is pronounced man-tis'sa.

CHARACTERISTIC OF A LOGARITHM

A characteristic may be either positive or negative. Its value depends entirely on the location of the decimal point in the original number. To get the characteristic for numbers which have figures to the left of the decimal point the following rule will apply:

Count the number of figures to the left of the decimal point and subtract 1 from this total and you have the characteristic of the number.

Some numbers are given in tabulated form in Table I to show you how this rule operates.

In the same way as shown in Table I, there can be any number of figures to the left of the decimal point and by subtracting one from the number of figures, you get the characteristic. These

LOGARITHMS

TABLE I

Number	Figures to left of Decimal Point	Minus	1	Equals	Positive Character- istic
2. 34567	1	—	1	=	0
23. 4567	2	—	1	=	1
234. 567	3	—	1	=	2
2345. 67	4	—	1	=	3
23456. 7	5	—	1	=	4

characteristics are all **positive**. Cover answers with a sheet of paper, then pick out the characteristics for the following numbers:

Number	Answer	Number	Answer
1. 23456	0	34,567,890.	7
654,321. 0	5	230,000. 00	5
897. 654	2	8,760,007. 2	6
37. 09843	1	92,007,000.	7
2768. 91	3	1. 0007239	0
7. 0	0	900.	2
100.	2	100,000,000.	8

When the figures are all to the **right** of the decimal point, the characteristic is **negative**. When the first figure to the **right** of the decimal point is **not a cipher**, the characteristic is a **minus one**

TABLE II

Number	Ciphers to right of Decimal Point	Plus	1	Equals	Negative Characteristic
. 234567	0	+	1	=	-1 or $\bar{1}$
. 023456	1	+	1	=	-2 or $\bar{2}$
. 002345	2	+	1	=	-3 or $\bar{3}$
. 0000234	4	+	1	=	-5 or $\bar{5}$
. 0000023	5	+	1	=	-6 or $\bar{6}$

(-1), regardless of the number of other figures to the right. When only the first figure to the right of the decimal point is a **cipher**, the

characteristic is a **minus 2** (-2), and for each **additional cipher** to the right of the decimal point add **one** to the next preceding characteristic and place a minus sign in front or above the result. This is illustrated in Table II.

In the same way as shown in Table II, there can be any number of ciphers to the **right of the decimal point**, and by adding one to the number of ciphers and placing a $-$ sign in front or above this sum, you get the characteristic. These characteristics are all **negative**. Cover answers with a sheet of paper, then pick out the characteristics for the following numbers:

Number	Answer	Number	Answer
0.3	$\bar{1}$	0.010203	$\bar{2}$
.0005	$\bar{4}$	.001002	$\bar{3}$
00.6789	$\bar{1}$	.0000009	$\bar{7}$
.00803	$\bar{3}$	00.0002003	$\bar{4}$
.00001	$\bar{5}$	000.000033	$\bar{5}$

The complete rule for obtaining either a positive or a negative characteristic may be stated as follows:

When the number is one or greater than one (1), the characteristic is positive and is one less than the number of figures to the left of the decimal point. When the number is less than one, the characteristic is negative and is one more than the number of ciphers between the decimal point and the first significant figure.

The characteristic of a number cannot be given in the tables because it depends entirely on the location of the decimal point.

MANTISSA OF A LOGARITHM

The mantissa is entirely independent of the decimal point. This is just the opposite of the characteristic. The mantissa is always positive and is always a decimal number. In other words it is the decimal part of a logarithm.

If you have two or more quantities composed of the same significant figures in the same order, it does not make any difference with the mantissa how many ciphers are to the right or to the left of the significant figures. The mantissa is the same in every case.

Let us study the Logarithmic Tables and find out just what is meant by the above statement. The following illustration is the first horizontal line on page 2 of the Logarithmic Tables.

LOGARITHMIC TABLES

N	0	1	2	3	4	5	6	7	8	9	D
100	000000	000434	000868	001301	001734	002166	002528	003029	003461	003891	432

The first vertical column in the Logarithmic Tables is headed by the letter **N**, which stands for the word **number**. This column contains the numbers whose mantissae are given in the horizontal lines corresponding with the numbers. Thus in the above illustration 100 is shown under **N** and the various mantissae for 100 with the following additional tenths (0, 1, 2, 3, 4, 5, 6, 7, 8, and 9) are shown in the horizontal line with 100 and in the vertical columns to correspond to the tenths given. To explain further, the mantissa for 100.0 is .000000 and the characteristic is 2, as shown on page 2 of the text. The mantissa for 100.1 is .000434, as found in the column headed by 1. The mantissa for 100.2 is .000868, as found in the column headed by 2. Thus as the tenths are added to 100, the mantissa increases until under the column headed by 9, the mantissa for 100.9 is given as .003891.

The equally spaced steps, one tenth apart, are for aiding the reader to obtain accurate results quickly. In the Logarithmic Tables the second number in column **N** is 101. If the intervening 10 steps between 100 and 101 were not given, it would take considerable time to calculate them.

As the numbers in column **N** progress to the bottom of the page, one unit is added as each line is passed. Thus from the first mantissa in column 0 to the last mantissa of column 9, a complete set of mantissae are given for each 1/10 interval for the numbers given in column **N**.

The Logarithmic Tables do not show any decimal points. As the mantissae are entirely decimal, the decimal point will be at the left of each left-hand figure.

In column **N** the decimal point can be placed at any place to suit the user. For instance, the first number may be 0.10, 1.00, 10.0,

100, 1000, 10000, or any other multiple of ten. The mantissa is the same in any case as the only significant figure is 1. The only difference is in the characteristic, as previously shown.

In the last column to the right in the Logarithmic Tables, headed **D**, the differences between any two adjoining mantissae in each horizontal line are given. Since there are ten mantissae in each line, the differences are not always the same; therefore, column **D** gives the average difference for each line. This variation does not make much difference ordinarily. This column is given to save you the labor of subtracting and also insures accurate work. The use for these differences as given in column **D** will be explained later.

HOW TO FIND THE LOGARITHM OF A NUMBER

From the accompanying illustration, copied from page 4 of the Logarithmic Tables, we will show you how to find the mantissa for any given number. Take number 220 in column **N**. Let us follow this line right across horizontally and see what the different mantissae are for 220, for 220.1, for 220.2, for 220.9, for 221, etc. See Table III.

LOGARITHMIC TABLES

N	0	1	2	9	D
220	.342423	.342620	.342817	.344196	197
221	.344392	.344589	.344785	.346157	196
222	.346353	.346549	.346744	.348110	195

TABLE III

Number	Mantissa	Column found in	Characteristic
220.0	.342423	0	2
220.1	.342620	1	2
220.2	.342817	2	2
220.9	.344196	9	2
221.0	.344392	0	2
222.2	.346744	2	2

In the vertical column under 0, first line, we find the mantissa .342423. This is in line with number 220 in column **N** and as there

are no tenths added in the **0** column, this is the proper mantissa for number 220.0. There are three figures to the left of the decimal point; by subtracting 1, according to Table I, we get the characteristic 2. Therefore, the complete logarithm for 220.0 is 2.342423.

In the next vertical column under **1**, first line, we find the mantissa .342620. This mantissa is larger than the one in column **0** in the same line because we have added one-tenth (0.1) to our number 220.0, making it 220.1. The characteristic has not changed, so our complete logarithm for 220.1 is 2.342620.

In the next vertical column under **2**, first line, we find the mantissa .342817. This has increased over the preceding mantissa because we have added one-tenth to our number 220.1, making it 220.2. The characteristic, however, remains 2 for we still have the same number of integers to the left of the decimal point in the number, so the logarithm for 220.2 is 2.342817.

As we progress to the right across the table we add 0.1 to the number shown in the next preceding column, and when we reach the column under **9**, the number becomes 220.9 and the mantissa given is .344196.

If we add 0.1 to this number 220.9, we have 221.0, which is the number given in the second line under **N**, and the mantissa is the second in the column under **0**.

Thus you see the table progresses in 0.1 steps from the number 220 until the bottom of the page is reached in the column under **9**. On this page 4 of the Logarithmic Tables, the last number under **N** is 250, so the mantissa for 250.9 is the last one in the column under **9**. Thus the complete logarithm for 250.9 is 2.399501. Therefore, to find the mantissa for any number from 200 to 250.9 you would use page 4 of the Logarithmic Tables.

These same numbers (220, 220.9, 221, etc.) may have the decimal point changed either to the right or to the left without changing the mantissae at all, but the characteristics will be changed. This is illustrated in Table IV.

In Table IV we have used the numbers as given in Table III, but have changed the decimal points in order to show you how the characteristics change but the mantissae are the same and do not change. Apply the rule which we gave you concerning the characteristic and refer to Tables I and II, and you will understand

TABLE IV

Number	Logarithm	Characteristic
2 200	0 342123	1 - 1 = 0
22 01	1 342620	2 - 1 = 1
220. 2	2 342817	3 - 1 = 2
2209	3 344196	4 - 1 = 3
0 221	$\bar{1}$ 344392	0 + (-1) = -1 or $\bar{1}$
0 0221	$\bar{2}$ 344392	-1 + (-1) = -2 or $\bar{2}$
.00221	3 344392	-2 + (-1) = -3 or $\bar{3}$

Table IV. To illustrate, note that the last three numbers in Table IV have the same consecutive figures (221), but the decimal point has been moved one figure more to the left in each case. Therefore, while these numbers have the same mantissa the characteristic has become negative and changed according to the number of ciphers to the right of the decimal point.

FINDING NUMBER THAT CORRESPONDS TO A LOGARITHM

In using the Logarithmic Tables it is also necessary to know how to find the number when the logarithm is given. Take, for example, logarithm 3.203848. To find the number which corresponds to this logarithm, we just reverse the process for finding the logarithm for the number.

Take the first three figures of the mantissa, which are 203, and turn to the Logarithmic Tables. Follow down the first vertical column 0 until we find nearly the same mantissa as the logarithm contains. Then follow across the horizontal line when necessary to get the same, or nearest to but less, mantissa than the one in the problem. We will find this mantissa in the tenth line of column 9, page 3. In this particular case, we find the exact mantissa which we have in our logarithm (Step 1).

The number which corresponds to this mantissa is given in the same line by the three figures in column N, which are 159, and we add on to the right end figure 9 from the column under 9. This gives the number 1599.

It is better to use a smaller mantissa than a larger one. This is shown later.

Step 1

LOGARITHMIC TABLES

Page	Line	N	9	D
3	Tenth	159	203848	272

Annex, or place, after 159 the figure 9 which makes the number 1599. $3+1=4$, so point off four figures to left of the decimal point.

To find where to point off the figures for the decimal point, we **reverse** the operations used in finding the characteristic. In our problem we have a characteristic of 3, to which we add 1, which makes 4. Begin at the left and count off four figures and place the decimal point.

Having a positive characteristic, the rule is: *Add one to the given characteristic to find the number of figures to the left of the decimal point.*

Having a negative characteristic, the rule is: *Subtract one from the negative characteristic to find the number of ciphers to the right of the decimal point.*

To illustrate these rules a number of logarithms and the way the corresponding numbers are pointed off for both the positive and the negative characteristic are given in Table V.

TABLE V Locating the Decimal Point

Logs. +Characteristic	To Characteristic add one	Figures to left of point	Number
0 .301030	0 + 1 = 1	One	2 00
1 .201120	1 + 1 = 2	Two	16 00
2 .161368	2 + 1 = 3	Three	115 00
3 .203848	3 + 1 = 4	Four	1599 00
5 .687013	5 + 1 = 6	Six	486121 4 (Prob 1)
Logs. -Characteristic	From characteristic subtract one	Ciphers to right of point	Number
1̄ .518511	1 - 1 = 0	None	.330
2̄ .521118	2 - 1 = 1	One	.0332
3̄ .653502	3 - 1 = 2	Two	.004503
4̄ .660865	4 - 1 = 3	Three	.000458
4̄ .331690	4 - 1 = 3	Three	.00021462 +

Let us now take a logarithm with a minus characteristic and follow it through in the same way as we did the logarithm with the positive characteristic. For example: We will take the last one in Table V. The logarithm has a minus 4 characteristic, and the mantissa is 331690. The minus sign is placed above the characteristic so that it will not be interpreted as a minus logarithm, for a minus sign in front of a number refers to all of the figures in the number following it. Take the first three figures of the mantissa, which are 331. Follow down through the column **0** on Page 4 of the Logarithmic Tables until we find the mantissa which contains the figures 330, which are the nearest to 331 in this column. In line 15, corresponding to number 214 in column **N**, we find these figures and by following this line over to vertical column **6**, we find the nearest mantissa to the one we have and a little less. You will note by looking **under** column **7** at the next mantissa that it is quite a little bit over (Step 2.)

Step 2

LOGARITHMIC TABLES

Page	Line	N	6	7	D
4	15	214	331630	331832	202

As before, we have three figures, 214, under the vertical column **N**, and to the right of that we annex figure 6 from column **6**, and that gives the four significant figures 2146. To this we must annex the difference of the mantissa in the table and our logarithm for the mantissa is a little less than the mantissa in the problem.

Step 3

$$\begin{array}{r}
 \bar{4} \ 331690 \text{ our logarithm} \\
 \quad \quad \quad 331630 \text{ our logarithm in the table} \\
 \text{Difference} = \underline{\quad \quad 60}
 \end{array}$$

In Step 3 we subtract the mantissa in column **6** from our logarithm. It gives us a difference of 60. We now have the question of how much to annex to the number 2146 for this difference of 60 in the mantissae. You will note the difference between the mantissae in columns **6** and **7** (Step 2) is the value given in column **D**, or 202. In other words, if we would use column **7** instead of column **6**,

the mantissa would have to be 202 more. As we have only 60 more than in column 6, we divide 60 by 202, which equals $29+$, or the fractional part which we annex to our number. This is shown in Step 4.

Step 4 $\frac{60}{202} = 29+$

Annexing $29+$ to 2146 we get 214629+

$4-1=3$ (Table V). Placing 3 ciphers to right of decimal point
 $= .000214629+$ Answer

MULTIPLICATION

In order to multiply two numbers by the use of logarithms, we find the logarithms of the two numbers and then add these two logarithms together. This sum is the logarithm of the product of the two numbers. Then we turn to the Logarithmic Tables and find the number which corresponds to this logarithm. We will take two of the numbers and their logarithms from Table III to illustrate.

NOTE: In problems use log, abbreviation for logarithm, and mant., abbreviation for mantissa.

Problem 1. Multiply 220.2 by 2209

<i>Instruction</i>	<i>Operation</i>
Step 1	Step 1
Find log of the first number in problem.	Find log of 220.2
Do this by finding the mantissa of the number.	Mantissa of 220.2 = .342817
Next find the characteristic of the number.	Characteristic of 220.2 = 2
Combine and you have the log of the number.	Log of 220.2 = <u>2.342817</u>
Step 2	Step 2
Find log of the second number in problem.	Find log of 2209
Do this by finding the mantissa of the number.	Mantissa of 2209 = .344196
Next find the characteristic of the number.	Characteristic of 2209 = 3.
Combine and you have the log of the number.	Log of 2209 = <u>3.344196</u>

Step 3

To multiply numbers, add the log of those numbers together.

Step 3

Add log of 220.2 and log of 2209 together

From Step 1, log of 220.2 = 2.342817

From Step 2, log of 2209 = 3.341196

Log of product = 5.687013

Step 4

Find number that corresponds to this logarithm

Do this by finding the number in Logarithmic Tables that has a mantissa the same as the mantissa in your problem or slightly less.

Subtract this mantissa from the given mantissa

Step 4

Find number that corresponds to given mantissa

From Step 3, mantissa = 687013

Mantissa of 4864 = 686994

Difference = 9

486 from column N

4 from column 4

4864 is four figures of number

LOGARITHMIC TABLES

Page	Line	N	4	5	D
9	37	486	686994	687085	89

Step 5

Find difference between mantissa in adjacent columns on same horizontal line in the log tables or use column D

Step 5

Mantissa of 4865 = 687085

Mantissa of 4864 = 686994

Difference = 89

Column D = 89

Step 6

Divide difference in Step 4 by difference in Step 5

Step 6

Divide 19 by 89

$$\begin{array}{r} 89 \overline{) 190} \quad 214 \\ \underline{178} \\ 120 \\ \underline{89} \\ 10 \end{array}$$

Step 7

Place the quotient of Step 6 at the right-hand end of the number found from the table, Step 4.

Step 7

Number in Step 4 = 4864

Quotient in Step 6 = 214

Result = 4864214

Step 8**Locate decimal point.**

Number of figures to left of decimal point is characteristic +1.

Begin at left of result in Step 7 and count six figures and place point.

Step 8

Characteristic of log is $5+1=6$ (Table V)

Decimal point is between figure 1 and figure 4

Thus 486421.4

Therefore 220.2 multiplied by 2209 is 486421.4

The problem will now be given in condensed form to show the actual work.

Multiply 220.2 by 2209.

Step 1	Log of 220.2	=	2.342817
Step 2	Log of 2209	=	3.344196
Step 3	Log of product	=	5.687013
Step 4	Mant. of 4864	=	.686994
	Difference	=	19
Step 5	Difference	=	89
Step 6	$19 \div 89$	=	.214
Step 7	214 annexed to 64	=	4864214
Step 8			486421.4 Answer

In order that you may understand all the details, we will describe this first problem carefully.

We have taken the two logs from Table II, set them down under each other, and added them and thus obtained the logarithm of a product. Now to find the number which corresponds to this logarithm of the product, we reverse the process for finding the logarithm. We turn to our Logarithmic Tables, and look for the mantissa which corresponds to the mantissa of the product.

To do this, take the first three figures of the mantissa of the product (in this case 687), and look for the same figures in a column in the Logarithmic Tables. You will find these figures on page 9 in the horizontal line where column N equals 486. By following across horizontally until you get to the vertical column 4, you will find the same mantissa which we have shown you in the solution, Step 4.

The figures 486 in column N are the first three figures of your new number, and the figure 4, at the head of column 4, is the fourth figure of your number.

These four figures we have placed in Step 4, in front of the mantissa which we have just found. This is the **nearest lower mantissa** to the one which we have in our logarithm for the product of the two numbers. We now subtract this mantissa from the one above it, as shown in Step 4. The difference is 19. This means that our number is a little larger than this mantissa, so now we must do what we call interpolate (pronounced in-tûr'pō-lāte). If you look in column D, Step 4 of our problem, you will find there is a difference between the mantissa in column 4 and the one in column 5 of 89. So we must divide the difference of the mantissae (19) by the difference (89) found in column D, to get the amount that we are to annex to the number 4864. This gives us the figures 214 to be annexed to our number, as shown in Step 7. We annex these three figures to the other four and we now have seven figures for our number.

To point off this number properly, we use the rule regarding the finding of the decimal point when given a positive characteristic. We have the characteristic of 5 in the logarithm of the product of the numbers shown in Step 3. To locate the decimal point, we begin at the left-hand figure and count off 5+1 figures, then place the decimal point as shown in Step 8. (See Table V.)

If you wish to check this problem to see whether it is right or not, you can multiply 2000 by 200, which are the round figures involved, and you will find there are six figures to the left of the decimal point. This proves that the problem is solved properly.

Now if you are to multiply the two numbers out in longhand to see whether it is correct or not, you will find that there is a slight difference in the decimal figures only. In other words your exact number would end in .8 instead of .4. You see your result is almost exactly what we obtained by the use of logs. Also you will notice that you have done very little actual figuring to get this product by logs.

In order to multiply 3, 4, 5, or 6 numbers together, all that is necessary is to add the logarithms of those numbers together, and finish the problem in the same way as Problem 1. Thus you see you can multiply 5 or 6 numbers together by the use of logarithms almost as easily as you can two. You would, of course, have the

extra logarithms to look up, otherwise the problem would be just the same as one operation for multiplication.

To show how easy it is to multiply several numbers together by logs, we will work out two problems in detail.

Problem 2. Multiply : $37 \times .46 \times .053 \times 7.9 \times 940$.

LOGARITHMIC TABLES

Page	N	0	Characteristic	Table
7	370	568202	2 - 1 = 1	I
9	460	662758	0 + 1 = 1	II
10	530	724276	1 + 1 = 2	II
15	790	897627	1 - 1 = 0	I
18	940	973128	3 - 1 = 2	I

Log of 37.	= 1.568202	} Add the mantissae column by column and hold the number to be carried over from the last column.
Log of .46	= 1.662758	
Log of .053	= 2.724276	
Log of 7.9	= 0.897627	
Log of 940.	= 2.973128	

Log of product	= 3.825991	} From the sum of the positive characteristics, plus what is carried over from adding the mantissae, subtract the sum of the minus characteristics which = +3+3-3 = +3 characteristic.

Mant. of 6698 = 825945 } Mantissa for first 4 figures of answer.

Difference of
mantissae = 46

LOGARITHMIC TABLES

Page	N	8	D
13	669	.825945	65

$46 \div 65 = 71 -$ (71 -) annexed to 6698 = 669871 - Characteristic $3+1=4$ Point off four figures, starting at the left, and it gives 6698.71 Answer

Problem 3. Multiply $250.9 \times 36.72 \times 9875 \times .04756$

LOGARITHMIC TABLES

Page	N	0	2	5	6	9	D
4	250					.399501	
7	367		564903				
19	987			994537			
9	475				677242		
8	432				636087		100

Log of 250.9	= 2.399501	Add logs for product	Characteristic				Table
Log of 36.72	= 1.564903		3	-	1	= 2	I
Log of .9875	= 1.994537		2	-	1	= 1	I
Log of .04756	= 2.677242		0	+	1	= 1	II
			1	+	1	= 2	II

Log of product = 2.636183 + 2 + 3 - 3 = 2 characteristic (like Problem 2).

Mant. of 4326 = 636087 = Mantissa for first 4 figures of answer.

Difference = 96

$96 \div 100 = 96$ 96 annexed to 4326 = 432696 Characteristic $2+1=3$

Point off three figures, starting at the left, and it gives 432.696 Answer.

These problems illustrate the small amount of figuring required to solve them by logs when compared to arithmetic.

In our previous problems in multiplication we had only four figures in each number. Now we will illustrate a problem in which there are six figures in each number. This problem will be worked out in detail in order that you may be able to follow it all the way through.

Problem 4. Multiply 369.875 by 4863.45*Instruction**Operation***Step 1**

Find log of first number in problem.

Do this by finding mantissa of first four figures of number; then find amount to be added for the other two figures. Next find the characteristic. Combine to get log of number.

Step 1

Find log of 369 875

Mantissa of 3698 = .567967
 Part added for 75 = 885
 Characteristic = 2
 Log of 369.875 = 2 5680555

.75 × 118 = 88 5 = added part

3 - 1 = 2 characteristic (Table I)

LOGARITHMIC TABLES

Page	N	8	9	D
7	369	567967	568084	118

Step 2

Find log of second number in problem.

Do this by finding mantissa of first four figures of number; then find amount to be added for the other two figures. Next find the characteristic.

Combine to get log of number.

Step 2

Find log of 4863 45

Mantissa of 4863 = 686904
 Part added for 45 = 40
 Characteristic = 3
 Log of 4863 45 = 3 686944

.45 × 89 = 40. + = added part

4 - 1 = 3 Characteristic (Table I)

LOGARITHMIC TABLES

Page	N	3	4	D
9	486	.686904	.686994	89

Step 3

To multiply numbers, add the logs of those numbers together.

Step 3

Add logs of 369.875 and 4863.45

Step 1 gives log 2.5680555
 Step 2 gives log 3.686944
 Log of product = 6.2549995

Step 4

Find number that corresponds to this logarithm.

Do this by finding the number in the Logarithmic Tables that has a mantissa the same as the mantissa of the product or slightly less.
Subtract the mantissae.

Step 4

Find the number that corresponds to mantissa of product

Step 3 mantissa . 2549995
Mantissa for 1798 . 254790
Difference 2095

From column N = 179
From column 8 = 8
First four figures = 1798

LOGARITHMIC TABLES

Page	N	8	9	D
3	179	. 254790	25.5031	242

Step 5

Find difference between mantissa in same horizontal line in the Logarithmic Tables, or use column D

Step 5

Mantissa of 1799 = . 255031
Mantissa of 1798 = . 254790
Column D = 242

Step 6

Find part to annex
Divide difference in Step 4 by difference in Step 5.

Step 6

Divide 2095 by 242

$$\begin{array}{r} 242 \overline{) 2095} \quad 865 + \\ \underline{1936} \\ 1590 \\ \underline{1452} \\ 1380 \\ \underline{1210} \\ 170 \end{array}$$

Step 7

Place the quotient at right-hand end of the number found from the Logarithmic Tables in Step 4.

Step 7

Number in Step 4 = 1798
Quotient in Step 6 = 865
1798865 +

Step 8

Locate decimal point.
Number of figures to left of decimal equals characteristic plus one.
Begin at left and count seven figures, then place point

Step 8

Characteristic of log of product : 6

6+1=7 (Table V)
Place decimal point at right end of the number 1798865. +
The product of 369.875 × 4863.45 = 1798865. + Answer

Below is shown the solution without explanation.

$$\begin{array}{lcl} \text{Step 1} & \text{Mant. of 3698} & = .567967 \\ & \text{Mant. of } 75 & = \underline{88.5} \\ & \text{Log of 369.875} & = 2.5680555 \end{array}$$

$$\begin{array}{lcl} \text{Step 2} & \text{Mant. of 4863} & = 686904 \\ & \text{Mant. of } 45 & = 40 \\ & \text{Log of 4863.45} & = 3.686944 \end{array}$$

$$\begin{array}{lcl} \text{Step 3} & \text{Log of 369.875} & = 2.5680555 \\ & \text{Log of product} & = \underline{6.2549995} \end{array}$$

$$\begin{array}{lcl} \text{Step 4} & \text{Mant. of 1798} & = .254790 \\ & \text{Difference} & = \underline{2095} \end{array}$$

$$\text{Step 5} \quad \text{Column D} \quad = \quad 242$$

$$\text{Step 6} \quad 2095 \div 242 = 865 +$$

$$\begin{array}{l} \text{Step 7} \quad 865 \text{ annexed to } 1798 = 1798865 + \\ \quad 1798865. + \text{ Answer} \end{array}$$

Description of Solution. We first find the logarithms of the numbers. We find the mantissa of the first four figures of the first number by turning to page 7, as illustrated in Step 1. Under column 8 we find the mantissa. We have now the two figures 7 and 5 to account for. The 75 is 75/100 of 1 unit in the column for figure 8, the fourth figure of our number 3698. In other words, if we would add 25 to 75 we would have 100 or 1 to add to 8. Thus, to get the amount which we are to add to this mantissa we have 75/100 of the difference between the mantissae in columns 8 and 9. Column D gives the difference 118, so we multiply 118 by .75, and point off accordingly, which gives 88.5. This is added to our mantissa, which gives the total mantissa for the whole number 369.875. We find the characteristic as shown in Table I.

We will now find the mantissa for the first four figures of the next number on page 9 of the Logarithmic Tables, as illustrated in Step 2. Under column 3 we find the proper mantissa. As in the first case, we have two figures left to find the amount we are to add on. This is found in the same way as before. This is added on in the same way and we find the characteristic in the same way as in Step 1.

Having our logarithms of the two numbers, we add them together in Step 3 for the log of the product, as we did in Problem 1. We have to find out what the corresponding number is for this logarithm as before. We will find the first four figures on page 3 of Logarithmic Tables as illustrated in Step 4. In column 8 we find the next less mantissa for the first four figures. This mantissa we subtract from the logarithm of the product and have a difference of 2095. This difference divided by the value in column D in Step 4 gives the value 865 (Step 6), which is the amount we annex to the four figures previously found (Step 7).

Next we place the decimal point in the right place by Table V. To the characteristic of 6 we add 1, which makes seven figures to the left of the decimal point. (Step 8.) This is the answer for the product of our two numbers.

While we have gone into quite a bit of detail to describe the operation to you, there is little actual figuring. When you compare this with the amount of time that would be required to multiply this out in the ordinary way, you will agree with us that logs can save you a lot of time besides eliminating the chances of errors.

Six-place tables are only accurate to sixth figure of answer.

Practice Problems

In order to give you practice in using the Logarithmic Tables, several practice problems in multiplication are given for you to solve by means of logarithms.

Follow the methods illustrated in the four problems already worked out. It is not necessary to send in the solutions of these practice problems unless you fail to get the answers given here. If you send them in, show all your work.

- | | |
|---|---|
| 1. Multiply $820 \times 40 \times 293$ | 9610422. Answer |
| 2. Multiply $423.1 \times 12.34 \times 65$ | 339367.9 Answer |
| 3. Multiply $9.99 \times .777 \times 67.8$ | 526.278 Answer |
| 4. Multiply $68.52 \times .0019 \times 371.8$ | 48.4039 Answer |
| 5. Multiply 254.76×4.3219 | 1101.05 Answer |
| 6. Multiply $.007342 \times .09837$ | .000722233 Answer |
| 7. Multiply $2700000 \times 39000000 \times 980000$ | 103,194,000,000,000,-
000,000 Answer |
| 8. Multiply 8100.07×3002.58 | 24,321,112 + Answer |

DIVISION

In order to divide two numbers by the use of logarithms, we find the logarithms of the two numbers and then subtract these two logarithms instead of adding them as we did in multiplication. To illustrate this, we will use the same two numbers we did in Problem 1 and divide them.

Problem 5. Divide 220.2 by 2209

*Instruction***Step 1**

Find log of the first number in problem.

Do this by finding the mantissa of the number. Next find the characteristic.

Combine the two and you have the log of the number.

Step 2

Find log of the second number.

Do this by finding the mantissa of the number. Next find the characteristic.

Combine the two and you have the log of the number.

Step 3

To divide one number by another subtract the logs of the numbers.

After 1 is borrowed from 2 to make 13 in the top line, instead of changing the 2 to 1, add 1 to the 3 in the bottom line, subtract the smaller number from the larger, and place a - sign in front of the difference:

$$+2 - (3 + 1) = -2 \text{ or } 2$$

Finding the Number that Corresponds to the Logarithm

Step 4

Find the first four figures of the number that corresponds to this logarithm.

Do this by finding the number in Logarithmic Tables that has a mantissa the same as the mantissa in your problem, or slightly less.

Subtract the mantissae.

*Operation***Step 1**

Find log of 220.2

Mantissa of 220.2 = .342817

Characteristic of 220.2 = 2.

Log of 220.2 = $\overline{2} \ 342817$

(See Table I)

Step 2

Find log of 2209

Mantissa of 2209 = .344196

Characteristic of 2209 = 3.

Log of 2209 = $\overline{3} \ 344196$

Step 3

Subtract log of 2209 from log of 220.2

From Step 1 log of 220.2 = 2.342817

From Step 2 log of 2209 = 3.344196

Log of quotient = $\overline{2} \ 998621$

Step 4

Find number that corresponds to mantissa.

From Step 3 log = 2.998621

Mantissa of 9968 = .998608

Difference = $\overline{13}$

LOGARITHMIC TABLES

Page	N	8	9	D
19	996	998608	998652	44

Figure 8 from Column 8 annexed to 996 = 9968.

Step 5

Find difference between mantissa in adjacent columns on same horizontal line in the Logarithmic Tables or use Column D.

Step 5

Mantissa of 9969	= .998652
Mantissa of 9968	= .998608
Difference	= 44
Column D	= 44

Step 6

Divide difference in Step 4 by difference in Step 5.

Step 6

Divide 13 by 44

$$\begin{array}{r} 44 \overline{) 13.0} \quad .29+ \\ \underline{88} \\ 420 \\ \underline{396} \\ 24 \end{array}$$

Step 7

Place the quotient at the right-hand end of the number found from the Table, Step 4.

Step 7

Number in Step 4	= 9968
Quotient in Step 6	= 29
	<u>996829</u>

Step 8

Locate decimal point.

Number of ciphers to right of decimal point is negative characteristic minus one, or -2 less $1 = -1$

Step 8

Characteristic of log is -2 less $1 = -1$

Decimal point is to left of one cipher = .0996829

Thus $220.2 \div 2209 = .0996829$ Ans.

Below we show how simple the problem is when the explanation is omitted.

Step 1	Log of 220.2	= 2.342817
Step 2	Log of 2209	= 3.344196
Step 3	Log of quotient	= 2.998621
Step 4	Mant. of 9968	= .998608
	Difference	= 13
Step 5	Difference	= 44
Step 6	$13 \div 44 = 29+$	
Step 7	29 annexed to 9968 = 996829	
Step 8	.0996829 Answer	

Since we are dividing one number by a larger one our result, of course, will be a decimal number. In that case, we will have a negative characteristic as described in Table II.

You notice that the larger characteristic is subtracted from the smaller one. In order to do this we need to add two to the smaller characteristic. Thus, we will have as a result a minus 2 as a characteristic for the difference.

We now proceed the same as we did with the problem in multiplication to find the corresponding number. We find the next mantissa nearest to the one of the quotient as shown in Step 4. Subtract these two mantissae and you have a difference of 13. Now turn over to column **D** where we find the difference 44, Step 5. We divide 13 by 44 as we did in our multiplication problem and we have the result 29. This is annexed to your number as indicated by the location of your mantissa in your table and we have the result in Step 7.

In order to point this off properly, refer to Table V. Here we have a negative characteristic of 2, and we subtract 1, which is just the reverse of finding the characteristic. That gives us one cipher to the right of the decimal, so point it off as in Step 8.

If you want to check this result, you can divide by the ordinary way and see how near our log method is correct. In actual figures, however, we have done very little mathematical work. It takes quite a little time to describe this log method to you, but after you learn it once, you will find it comes easy.

Practice Problems

Solve the following problems by logarithms. Do not send in the solutions unless you fail to get the answers given. If you send them in, show all your work.

- | | |
|------------------------------|----------------|
| 1. Divide 847.6 by 24.99 | 33.9176 Answer |
| 2. Divide 7256.2 by 879.26 | 8.2526 Answer |
| 3. Divide 276.543 by 912.34 | .303113 Answer |
| 4. Divide 0.9783 by .1234 | 7.92787 Answer |
| 5. Divide .00879 by .0092 | .955435 Answer |
| 6. Divide 8321000 by 2135000 | 3.89743 Answer |
| 7. Divide 850.06 by 50.082 | 16.9733 Answer |
| 8. Divide .00097 by .0023 | .421739 Answer |

You will find that the knowledge of logarithms will come in very handy when you take up your advanced studies. In these texts there are a number of problems in which four, five, or six numbers are multiplied together and sometimes one or two divided. Both of these operations can be done at the same time by logarithms. In other words, you can multiply and divide at the same time. We will illustrate by a problem. Let us use three of the numbers, of which we already have the logarithms in our first table.

MULTIPLICATION AND DIVISION

Problem 6. Find the result $\frac{220 \times 22 \times 01}{2209}$

Step 1

$$\begin{array}{rcl}
 \text{Log of } 220 & = & 2.342817 \\
 \text{Log of } 22 & = & 1.342620 \\
 \text{Log of product} & = & 3.685437 \\
 \text{Log of } 2209 & = & 3.344196 \\
 \text{Log of quotient} & = & 0.341241 \\
 \text{Mant. of } 2194, & & \\
 \text{Step 2} & = & \frac{341237}{4} \\
 \text{Difference} & = & \frac{341237}{4}
 \end{array}$$

Add logs for product of number
Subtract logs for quotient of numbers

Step 2

LOGARITHMIC TABLES

Page	N	4	5	D
4	219	341237	341435	198

Step 3

$$4 \div 198 = 02$$

$$\text{Annexing } 02 \text{ to } 2194 = 219402$$

Step 4

Characteristic $0+1=1$ Point off one figure to left of decimal
(Table V) 2.19402 Answer

In this problem, add the first two logarithms. This gives the logarithm of the product. Subtract the logarithm of the third num-

ber from the logarithm of the product. This gives the quotient. Now all we have to do is to find the corresponding number for this logarithm. In the Logarithmic Tables, as shown in Step 2, you will find the next mantissa nearest to the one we have for the quotient. The difference is 4. We divide the difference by the difference in column D, Step 2, which is 198. The result is 02. We annex 02 to the right-hand end of our number 2194 and we get 219402, which is the whole number we are looking for. To find where to put the decimal point, we refer to Table V. Having a zero characteristic, we add one and this gives us one figure to the left of the decimal.

Thus you see we finished these two operations almost as quickly as we did the one before. The only difference is the extra step of subtracting one of the logarithms from the sum of the other two. Thus you see the use of logarithms will save you quite a bit of time from what is necessary to work this kind of a problem by arithmetic.

Practice Problems

Solve the following problems by logarithms. Do not send in the solutions unless you fail to get the answers given. If you send them in, show all your work.

1. $\frac{67432 \times .02035}{700.09}$ 1.96009 Answer
2. $\frac{25.729 \times 1.0025}{35.678}$.722947 Answer
3. $\frac{29000 \times 670000 \times 5300}{80029}$ 1,286,770,000 Answer

ROOTS AND POWERS

In the introductory discussion and in the lesson on Powers and Roots we explained the ordinary use of the square root or radical sign and exponents. This same sign is often used for other roots beside the square root. For instance, $\sqrt[3]{7235}$ means that the cube root of the number under the radical sign is to be obtained. We may also use any other figure in place of the 3 to indicate any root of the number under the radical sign. $\sqrt[4]{256}$ indicates that the fourth

root of 256 is to be obtained. The result equals 4, for $4 \times 4 \times 4 \times 4 = 256$. Also, $\sqrt[5]{243}$ indicates the fifth root of 243, which equals 3. Or $\sqrt[6]{64}$ indicates the sixth root of 64, which equals 2.

Ordinarily, there is a single figure used as an exponent for a number. The exponent indicates the power to which the number is to be increased. These exponents can be either fractional or decimal. $853^{2/3}$ is an illustration of a fractional exponent. It is read 853 to the 2/3rds power. This indicates that we are to raise 853 to the second power and extract the cube root. Thus, the numerator of the exponent indicates the power and the denominator indicates the root. $275^{7/5}$ is read 275 to the 7/5ths power. In this exponent the numerator is larger than the denominator. Thus it is seen that any kind of a fraction may be used as an exponent.

$724^{1.6}$ is an illustration of a decimal exponent. It is read 724 to the one and six-tenths power. This type of exponent may be either part or wholly decimal.

We have a great variety of exponents, but we can solve all problems containing these indicated operations by logarithms. Some of these operations are impossible by arithmetic. Therefore, you see how valuable logarithms are in this particular work.

In order to solve a problem in which we want to find the root, we find the logarithm of the number and then divide the logarithm by the exponent of the root of the number. For instance, if we want the square root of 75, we find the logarithm of 75 and divide the log by 2, which gives us the logarithm of the square root of the number. To find the cube root, we divide by 3; to find the fourth root, we divide by 4, etc.

To find the power of a number, we find the logarithm of the number and multiply it by the exponent of the power to which we want to raise the number. This gives the log of the power. You will note that this is just the reverse of finding the root.

For instance, if we want to find the square of 229, we find the logarithm of 229, and then multiply it by 2. That gives us the logarithm of the square of the number. If we multiply the logarithm of any given number by any indicated power, we obtain the logarithm of that power for that number. Fractional or decimal exponents are handled in a similar manner. We will work out two problems to illustrate how this work is done.

Problem 7. Solve $99^{9/7}$ **Step 1**

Log of 99	=	1.995635	—Logarithmic Tables, page 19
		<u>9</u>	Multiply by 9 to obtain the
		7)17.960715	log of 9th power of number
			Divide by 7 to obtain the log
			of the 7th root
Log of $99^{9/7}$	=	2.5658164+	
Mant. of 3679	=	<u>5657.30</u>	
Difference	=	864	

Step 2

LOGARITHMIC TABLES

Page	N	9	D
7	367	5657.30	118

Step 3

$$864 \div 118 = 7.3$$

$$7.3 \text{ annexed to } 3679 = 3679.73$$

Step 4

$$\text{Characteristic } 2 + 1 = 3$$

Point off three figures to the left of the decimal point.

367.973 Answer

First, find the mantissa for 99. This is easily located on Page 19 of Logarithmic Tables. The characteristic is located by Table I.

To get the ninth power of this logarithm, you multiply it by 9; and to get the seventh root of that, you divide it by 7. That gives you the logarithm of the two operations. We next find the corresponding number to this log as we have done in the other problems.

In this problem you will see you have saved yourself a lot of time in working it out with logs. You can find the ninth power by arithmetic, but you could not find the seventh root that way.

Problem 8. Solve $741.4^{1.6}$

Step 1

Log of 741.4 = 2.870053 Log Table, Page 14. Characteristic, Table I.

Multiply 1.6 = $\frac{1.6}{\quad}$ Multiply by exponent to get the log of the power.

17220318

2870053

Log of power = 4.5920848

Mant. of 3909 = $\frac{592066}{\quad}$ See Step 2

Difference = $\frac{188}{\quad}$

Step 2

LOGARITHMIC TABLES

Page	N	9	D
7	390	.592066	111

Step 3

$188 \div 111 = 17 -$

$17 -$ annexed to 3909 = 390917 -

Step 4

Characteristic $4 + 1 = 5$

Point off five figures to left of decimal point, Table V.

39091.7 Answer

We find the logarithm of 741.4 in the same way as we did in the other problems, then multiply by 1.6. That gives us the logarithm of the 741.4 to the 1.6 power. We find the nearest less mantissa in the Logarithmic Tables and subtract it as shown. Steps 3 and 4 are the same as in the other problems. This problem cannot be solved by arithmetic.

Practice Problems

Work out the following problems by logarithms. Do not send in the solutions unless you fail to get the given answers. If you send them in, show all of your work.

1. Find the value of $\sqrt[5]{476.92}$ 3.43312 Answer

2. Find the value of $12345^{9/3}$ 1,881,365,200,000 + Answer

3. Find the value of $9.645^{3.5}$ 2786.48 Answer

DECIMAL NUMBERS WITH EXPONENTS

Decimal numbers have negative characteristics, as we have shown in previous discussions. When we find it necessary to solve problems having decimal numbers with **powers** or **roots**, it will be necessary to change the negative characteristic to a positive characteristic.

As an example let us find the log of .07854³

Step 1

Log of .07854³ = $\overline{2}.895091 \times 3$ Table II

In order to avoid negative characteristics, we can make them positive by adding 10 to the characteristics and subtracting 10 at the end of the logarithms. This is illustrated in Step 2.

$$\begin{array}{r} \text{Step 2} \quad \overline{2}.895091 \times 3 \\ +10 \quad \quad -10 \\ \hline (8.895091 - 10) \times 3 \end{array}$$

Multiplying as indicated, gives us Step 3. We can simplify, by subtracting 20 from each part of the characteristic as shown.

$$\begin{array}{rcl} \text{Step 3} & 26.685273 - 30 & 26 - 30 = \overline{4} \\ & \underline{20 \quad \quad 20} & \\ & 6.685273 - 10 & +6 - 10 = \overline{4} \end{array}$$

Now we can go back to the negative form by subtracting the 10 from the 6, which gives us a minus 4 characteristic with the mantissa shown in Step 4.

$$\text{Step 4} \quad \overline{4}.685273 \text{ Answer}$$

This is the correct result, which cannot be obtained by multiplying directly the negative characteristics shown in Step 1.

Still another complication arises when you have the root instead of the power of a decimal number.

As an example let us find the log of $\sqrt[4]{.07854}$

This is the same number as used in the above problem, so we have the same logarithm and the negative characteristic, Steps 1 to 4.

Step 5 $(\bar{2}.895091 \div 4)$

In this case we have to divide by 4 in order to get the log of the root. Change the negative to the positive characteristic and we have Step 6.

Step 6 $(8.895091 - 10) \div 4$

In dividing such a logarithm we must have a -10 at the end after the division. Therefore, we must add to both parts of the characteristic to allow for this condition. In this case we add 30 as shown in Step 7. Dividing as indicated we get Step 8.

Step 7

$(8.895091 - 10) \div 4$
30 -30
<u>4)38.895091 - 40</u>

Step 8 $9.72377275 - 10$

Now we can put this back in a negative form, as before, and $9 - 10$ gives us -1 for a characteristic, as shown in Step 9. This is the correct logarithm for our problem.

Step 9 $\bar{1}.72377275$ Answer

This is the only method by which problems of this kind can be solved. We will solve a problem of this type to show you how it is done.

Problem 9. Solve: $\frac{(.07345)^3}{(.62547)^{1/2}}$

Log of .07345 = $\bar{2}.865992$ Logarithmic Tables, Page 14. Characteristic, Table II.

Change to positive characteristic = $8.865992 - 10$ ($8 - 10 = -2$)

Multiply by 3 for log of power = $\frac{3}{26.597976 - 30}$

This gives log of numerator

Mant. of 6254 = $.796158$ Logarithmic Tables, Page 12

Add for fifth figure 7 = $\frac{48.3}{1.7962063}$ ($7 \times 69 = 48.3$)

Log of .62547 = 1.7962063 Characteristic, Table II

Change to + characteristic = $9.7962063 - 10$ ($9 - 10 = -1$)

Add 10 to both parts = $\frac{10}{-10}$

Divide by 2 for root = $\frac{2}{19.7962063 - 20}$

This gives log of denominator = $9.89810315 - 10$

Subtract log of denominator from log of numerator in order to divide the numbers.

$$\text{Log of numerator} = 26.597976 - 30$$

$$\text{Log of denominator} = 9.89810315 - 10$$

$$\text{Log of quotient} = 16.69987285 - 20$$

$$\text{Change to negative characteristic} = 4.69987285 - (+16 - 20 = -4)$$

$$\text{Mant. of 5010} = .699838$$

$$\text{Difference} = 3485$$

LOGARITHMIC TABLES

Page	N	0	D
10	501	699838	87

Divide problem difference by column **D** value. $3485 \div 87 = 40$

Annex to other figures at right-hand end. Annexing 4 to 5010 = 50104

Table V, $4 - 1 = 3$

There will be three ciphers to right of decimal point.

.00050104 Answer

COMBINATION OF OPERATIONS

We are going to work out in detail a problem, which will illustrate four or five of the different operations at once. It is only a little different from the first problem illustrated, but is more complicated. The two numbers in the numerator both have simple exponents, while one number in the denominator has a fractional exponent and the other has a decimal exponent. And the radical is for the cube root of the total result of the other operations.

We have worked out this problem, giving the details of each operation so that you can follow it all the way through. Study this very carefully as it illustrates many of the different operations at once. It also shows you how much time and labor can be saved by using logarithms and how the chance for many mistakes is eliminated. Some of these operations cannot be solved by arithmetic.

After you have studied this problem through thoroughly so

that you understand it, you should be able to handle any kind of a problem which involves logarithms.

You must remember that you cannot add or subtract numbers after you get the logarithms of them. You must add or subtract by the ordinary methods before you get the logs. When you add the logarithms, you multiply the numbers; and when you subtract the logarithms, you divide the numbers.

Problem 10. Find value of: $\sqrt[3]{\frac{3678^4 \times 3.257^2}{1679^2 \times 1.345^{1.5}}}$

Step 1

Find log of first number of denominator.

Do this by finding mantissa of number in Logarithmic Tables, page 3.

Next find characteristic of number and combine.

Next find log for power of number.

Next find log for root of number

Step 1

Find log of $1679^{2/3}$

Mantissa of 1679 = .225051

Characteristic, Table

I = 3

Log of 1679 = 3.225051

Multiply by power 2

Divide by root 3.6 450102

Log of $1679^{2/3}$ = 2.150034

Step 2

Find log of second number of denominator.

Do this by finding mantissa of number in Logarithmic Tables, page 2.

Next find characteristic of number and combine.

Next find log for power of number.

Step 2

Find log of $1.345^{1.5}$

Mantissa of 1.345 = .128722

Characteristic, Table

I = 0

Log of 1.345 = 0.128722

Multiply by power 1.5

643610

128722

Log of $1.345^{1.5}$ = 0.193083

Step 3

Find log of denominator.

Do this by adding the two logs of the numbers for log of product of numbers.

Step 3

Add logs of $1679^{2/3}$ and $1.345^{1.5}$

Step 1 gives log 2.150034

Step 2 gives log 0.193083

Log of denominator = 2.343117

Step 4

Find log of first number of numerator.

Do this by finding mantissa of number in Logarithmic Tables, page 7

Next find characteristic of number and combine.

Next find log for power of number.

Step 4

Find log of 3678^4

Mantissa of 3678 = .565612

Characteristic,

Table I = 3

Log of 3678 = 3.565612

Multiply by power 4

Log of 3678^4 = 14.262448

Step 5

Find log of second number of numerator.

Do this by finding mantissa of number in Logarithmic Tables, page 6.

Next find characteristic of number and combine.

Next find log for power of number.

Step 5

Find log of 3257^2

Mantissa of 3257 = .512818

Characteristic,

Table I = 0.

Log of 3257 = 0.512818

Multiply by power 2

Log of 3257^2 = 1.025636

Step 6

Find log of numerator.

Do this by adding the logs of the two numbers for log of product.

Step 6

Add logs of 3678^4 and 3.257^2

Step 4, log of 3678^4 = 14.262448

Step 5, log of 3.257^2 = 1.025636

Log of numerator = 15.288084

Step 7

Find log of all numbers under the radical sign.

Do this by subtracting log of denominator from log of numerator for quotient of numerator divided by denominator.

Step 7

Subtract log of denominator from log of numerator

Step 6, log of numerator = 15.288084

Step 3, log of denominator = 2.343117

Log of quotient = 12.944967

Step 8

Find root of combined results under radical sign.

Do this by dividing log of quotient of combined numerator and denominator by indicated root of the radical sign.

Step 8

Find cube root of number whose log by Step 7 = 12.944967

Divide by 3 3)12.944967

Log of cube root = 4.314989

Step 9

Find number that corresponds to log of Step 8.

Do this by finding the number in Logarithmic Tables that has a mantissa the same or a little less than mantissa of the log.

Subtract mantissa.

Step 9

Find number that corresponds to mantissa of Step 8 = .314989

Column N gives 206

Column 5 gives 5
2065

Mantissa for 2065 = .314920

Difference 69

LOGARITHMIC TABLES

Page	N	5	D
4	206	314920	210

Step 10

Find amount to be annexed to the number from the difference of mantissa of Step 9.

Do this by dividing this difference by the difference shown in column D.

Step 10

Divide 69 by 210

$$\begin{array}{r} 210 \overline{) 690} \quad 328 + \\ \underline{630} \\ 600 \\ \underline{420} \\ 1800 \\ \underline{1680} \end{array}$$

Step 11

Place quotient of Step 10 at right end of number found in Step 9 to get all figures of resulting number.

Step 11

$$\begin{array}{rcl} \text{Number in Step 9} & = & 2065 \\ \text{Quotient in Step 10} & = & \frac{328}{2065328} \end{array}$$

Step 12

Locate decimal point.

Number of figures to left of decimal point equals characteristic plus one.

Step 12

$$\begin{array}{rcl} \text{Step 8 Characteristic} & = & 4 \\ \text{Add one} & & \frac{1}{5} \end{array}$$

Count five figures from left end and place decimal point.

20653.28 Answer

N.	0	1	2	3	4	5	6	7	8	9	D.
100	00 0000	00 0434	00 0868	00 1301	00 1734	00 2166	00 2598	00 3029	00 3461	00 3891	432
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	425
102	8600	9026	9451	9876	01 0300	01 0724	01 1147	01 1570	01 1993	01 2415	424
103	01 2837	01 3259	01 3680	01 4100	4521	4940	5360	5779	6197	6616	420
104	7033	7451	7868	8284	8700	9116	9532	9947	02 0361	02 0775	416
105	02 1189	02 1603	02 2016	02 2428	02 2841	02 3252	02 3664	02 4075	02 4486	02 4896	412
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
107	9384	9789	03 0195	03 0600	03 1004	03 1408	03 1812	03 2216	03 2619	03 3021	404
108	03 3424	03 3826	4227	4628	5029	5430	5830	6230	6629	7028	400
109	7426	7825	8223	8620	9017	9414	9811	04 0207	04 0602	04 0998	397
110	04 1393	04 1787	04 2182	04 2576	04 2969	04 3362	04 3755	04 4148	04 4540	04 4932	393
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	390
112	9218	9606	9993	05 0380	05 0766	05 1153	05 1538	05 1924	05 2309	05 2694	386
113	05 3078	05 3463	05 3846	4230	4613	4996	5378	5760	6142	6524	383
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	06 0320	379
115	06 0698	06 1075	06 1452	06 1829	06 2206	06 2582	06 2958	06 3333	06 3709	06 4083	376
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	373
117	8186	8557	8928	9298	9668	07 0038	07 0407	07 0776	07 1145	07 1514	370
118	07 1882	07 2250	07 2617	07 2985	07 3352	3718	4085	4451	4816	5182	366
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	07 9181	07 9543	07 9904	08 0266	08 0626	08 0987	08 1347	08 1707	08 2067	08 2426	360
121	08 2785	08 3144	08 3503	3861	4219	4576	4934	5291	5647	6004	357
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
123	9905	09 0258	09 0611	09 0963	09 1315	09 1667	09 2018	09 2370	09 2721	09 3071	352
124	09 3422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	09 6910	09 7257	09 7604	09 7951	09 8298	09 8644	09 8990	09 9335	09 9681	10 0026	346
126	10 0371	10 0715	10 1059	10 1403	10 1747	10 2091	10 2434	10 2777	10 3119	3462	343
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	341
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	11 0253	338
129	11 0590	11 0926	11 1263	11 1599	11 1934	11 2270	11 2605	11 2940	11 3275	3609	335
130	11 3943	11 4277	11 4611	11 4944	11 5278	11 5611	11 5943	11 6276	11 6608	11 6940	333
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	12 0245	330
132	12 0574	12 0903	12 1231	12 1560	12 1888	12 2216	12 2544	12 2871	12 3198	3525	328
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	13 0012	323
135	13 0334	13 0655	13 0977	13 1298	13 1619	13 1939	13 2260	13 2580	13 2900	13 3219	321
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	316
138	9879	14 0194	14 0508	14 0822	14 1136	14 1450	14 1763	14 2076	14 2389	14 2702	314
139	14 3015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	14 6128	14 6438	14 6748	14 7058	14 7367	14 7676	14 7985	14 8294	14 8603	14 8911	309
141	9219	9527	9835	15 0142	15 0449	15 0756	15 1063	15 1370	15 1676	15 1982	307
142	15 2288	15 2594	15 2900	3205	3510	3815	4120	4424	4728	5032	305
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
144	8362	8664	8965	9266	9567	9868	16 0168	16 0469	16 0769	16 1068	301
145	16 1368	16 1667	16 1967	16 2266	16 2564	16 2863	16 3161	16 3460	16 3758	16 4055	299
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
148	17 0262	17 0555	17 0848	17 1141	17 1434	17 1726	17 2019	17 2311	17 2603	17 2895	293
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
150	17 6091	17 6381	17 6670	17 6959	17 7248	17 7536	17 7825	17 8113	17 8401	17 8689	289
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
150	17 6091	17 6381	17 6670	17 6959	17 7248	17 7536	17 7825	17 8113	17 8401	17 8689	289
151	8977	9264	9552	9839	18 0126	18 0413	18 0699	18 0986	18 1272	18 1558	287
152	18 1844	18 2129	18 2415	18 2700	2985	3270	3555	3839	4123	4407	285
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	19 0051	281
155	19 0332	19 0612	19 0892	19 1171	19 1451	19 1730	19 2010	19 2289	19 2567	19 2846	279
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
157	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
158	8657	8932	9206	9481	9755	20 0029	20 0303	20 0577	20 0850	20 1124	274
159	20 1397	20 1670	20 1943	20 2216	20 2488	2761	3033	3305	3577	3848	272
160	20 4120	20 4391	20 4663	20 4934	20 5204	20 5475	20 5746	20 6016	20 6286	20 6556	271
161	6826	7096	7365	7631	7904	8173	8441	8710	8979	9247	269
162	9515	9783	21 0051	21 0319	21 0586	21 0853	21 1121	21 1388	21 1651	21 1921	267
163	21 2188	21 2451	2720	2986	3252	3518	3783	4049	4311	4579	266
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
165	21 7481	21 7747	21 8010	21 8273	21 8536	21 8798	21 9060	21 9323	21 9585	21 9846	262
166	22 0108	22 0370	22 0631	22 0892	22 1153	22 1414	22 1675	22 1936	22 2196	22 2456	261
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7629	258
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	23 0193	256
170	23 0449	23 0704	23 0960	23 1215	23 1470	23 1724	23 1979	23 2234	23 2488	23 2742	255
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
173	8046	8297	8548	8799	9049	9299	9550	9800	24 0050	24 0300	250
174	24 0549	24 0799	24 1048	24 1297	24 1546	24 1795	24 2044	24 2293	2541	2790	249
175	24 3038	24 3286	24 3534	24 3782	24 4030	24 4277	24 4525	24 4772	24 5019	24 5266	248
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	216
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	25 0176	245
178	25 0420	25 0664	25 0908	25 1151	25 1395	25 1638	25 1881	25 2125	25 2368	2610	243
179	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	25 5273	25 5514	25 5755	25 5996	25 6237	25 6477	25 6718	25 6958	25 7198	25 7439	241
181	7679	7918	8158	8398	8637	8877	9116	9355	9591	9833	239
182	26 0071	26 0310	26 0548	26 0787	26 1025	26 1263	26 1501	26 1739	26 1976	26 2214	238
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
185	26 7172	26 7406	26 7641	26 7875	26 8110	26 8344	26 8578	26 8812	26 9046	26 9279	234
186	9513	9746	9980	27 0213	27 0446	27 0679	27 0912	27 1144	27 1377	27 1609	233
187	27 1842	27 2074	27 2306	2538	2770	3001	3233	3464	3696	3927	232
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
190	27 8754	27 8982	27 9211	27 9439	27 9667	27 9895	28 0123	28 0351	28 0578	28 0806	228
191	28 1033	28 1261	28 1488	28 1715	28 1942	28 2169	2396	2622	2849	3075	227
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
195	29 0035	29 0257	29 0480	29 0702	29 0925	29 1147	29 1369	29 1591	29 1813	29 2034	222
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
199	8853	9071	9289	9507	9725	9943	30 0161	30 0378	30 0595	30 0813	218
200	30 1030	30 1247	30 1464	30 1681	30 1898	30 2114	30 2331	30 2547	30 2764	30 2980	217
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
200	30 1030	30 1247	30 1464	30 1681	30 1898	30 2114	30 2331	30 2547	30 2764	30 2980	217
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
204	9630	9843	31 0056	31 0268	31 0481	31 0693	31 0906	31 1118	31 1330	31 1542	212
205	31 1754	31 1966	31 2177	31 2389	31 2600	31 2812	31 3023	31 3234	31 3445	31 3656	211
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
209	32 0146	32 0354	32 0562	32 0769	32 0977	32 1184	32 1391	32 1598	32 1805	32 2012	207
210	32 2219	32 2426	32 2633	32 2839	32 3046	32 3252	32 3458	32 3665	32 3871	32 4077	206
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
213	8380	8583	8787	8991	9194	9398	9601	9805	33 0008	33 0211	203
214	33 0414	33 0617	33 0819	33 1022	33 1225	33 1427	33 1630	33 1832	2034	2236	202
215	33 2438	33 2640	33 2842	33 3044	33 3246	33 3447	33 3649	33 3850	33 4051	33 4253	202
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
218	8456	8656	8855	9054	9253	9451	9650	9849	34 0047	34 0246	199
219	34 0444	34 0642	34 0841	34 1039	34 1237	34 1435	34 1632	34 1830	2028	2225	198
220	34 2423	34 2620	34 2817	34 3014	34 3212	34 3409	34 3606	34 3802	34 3999	34 4196	197
221	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	196
222	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	195
223	8305	8500	8694	8889	9083	9278	9472	9666	9860	35 0054	194
224	35 0248	35 0442	35 0636	35 0829	35 1023	35 1216	35 1410	35 1603	35 1796	1989	193
225	35 2183	35 2375	35 2568	35 2761	35 2954	35 3147	35 3339	35 3532	35 3724	35 3916	193
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
228	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	190
229	9835	36 0025	36 0215	36 0404	36 0593	36 0783	36 0972	36 1161	36 1350	36 1539	189
230	36 1728	36 1917	36 2105	36 2294	36 2482	36 2671	36 2859	36 3048	36 3236	36 3424	188
231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
234	9216	9401	9587	9772	9958	37 0143	37 0328	37 0513	37 0698	37 0883	185
235	37 1068	37 1253	37 1437	37 1622	37 1806	37 1991	37 2175	37 2360	37 2544	37 2728	184
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
239	8398	8580	8761	8943	9124	9306	9487	9668	9849	38 0030	181
240	38 0211	38 0392	38 0573	38 0754	38 0934	38 1115	38 1296	38 1476	38 1656	38 1837	181
241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	180
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
244	7390	7568	7746	7924	8101	8279	8456	8634	8811	8989	178
245	38 9166	38 9343	38 9520	38 9698	38 9875	39 0051	39 0228	39 0405	39 0582	39 0759	177
246	39 0935	39 1112	39 1288	39 1464	39 1641	1817	1993	2169	2345	2521	176
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
249	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174
250	39 7940	39 8114	39 8287	39 8461	39 8634	39 8808	39 8981	39 9154	39 9328	39 9501	173
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
250	39 7940	39 8114	39 8287	39 8461	39 8634	39 8808	39 8981	39 9154	39 9328	39 9501	173
251	9674	9847	40 0020	40 0192	40 0365	40 0538	40 0711	40 0883	40 1056	40 1228	173
252	40 1401	40 1573	1745	1917	2089	2261	2433	2605	2777	2949	172
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
255	40 6540	40 6710	40 6881	40 7051	40 7221	40 7391	40 7561	40 7731	40 7901	40 8070	170
256	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	169
257	9933	41 0102	41 0271	41 0440	41 0609	41 0777	41 0946	41 1114	41 1283	41 1451	169
258	41 1620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
259	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	167
260	41 4973	41 5140	41 5307	41 5474	41 5641	41 5808	41 5974	41 6141	41 6308	41 6474	167
261	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166
262	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	165
263	9956	42 0121	42 0286	42 0451	42 0616	42 0781	42 0945	42 1110	42 1275	42 1439	165
264	42 1604	1768	1933	2097	2261	2426	2590	2754	2918	3082	164
265	42 3246	42 3410	42 3574	42 3737	42 3901	42 4065	42 4228	42 4392	42 4555	42 4718	164
266	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	163
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
268	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	162
269	9752	9914	43 0075	43 0236	43 0398	43 0559	43 0720	43 0881	43 1042	43 1203	161
270	43 1364	43 1525	43 1685	43 1846	43 2007	43 2167	43 2328	43 2488	43 2649	43 2809	161
271	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409	160
272	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	159
273	6163	6322	6481	6640	6799	6957	7116	7275	7433	7592	159
274	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	158
275	43 9333	43 9491	43 9648	43 9806	43 9964	44 0122	44 0279	44 0437	44 0594	44 0752	158
276	44 0909	44 1066	44 1224	44 1381	44 1538	1695	1852	2009	2166	2323	157
277	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	157
278	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	156
279	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003	155
280	44 7158	44 7313	44 7468	44 7623	44 7778	44 7933	44 8088	44 8242	44 8397	44 8552	155
281	8706	8861	9015	9170	9324	9478	9633	9787	9941	45 0095	154
282	45 0249	45 0403	45 0557	45 0711	45 0865	45 1018	45 1172	45 1326	45 1479	1633	154
283	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	153
285	45 4845	45 4997	45 5150	45 5302	45 5454	45 5606	45 5758	45 5910	45 6062	45 6214	152
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	152
287	7782	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
288	9392	9543	9694	9845	9995	46 0146	46 0296	46 0447	46 0597	46 0748	151
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291	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	149
292	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	149
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366	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548	119
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373	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	116
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446	9335	9432	9530	9627	9724	9821	9919	65 0016	65 0113	65 0210	97
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472	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769	92
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488	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	89
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490	69 0196	69 0285	69 0373	69 0462	69 0550	69 0639	69 0728	69 0816	69 0905	69 0993	89
491	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877	88
492	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	88
493	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639	88
494	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	88
495	69 4605	69 4693	69 4781	69 4868	69 4956	69 5044	69 5131	69 5219	69 5307	69 5394	88
496	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269	87
497	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	87
498	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	87
499	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883	87
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508	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	85
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514	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723	84
515	71 1807	71 1892	71 1976	71 2060	71 2144	71 2229	71 2313	71 2397	71 2481	71 2566	84
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531	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	82
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533	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	81
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541	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919	80
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544	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317	80
545	73 6397	73 6476	73 6556	73 6635	73 6715	73 6795	73 6874	73 6954	73 7034	73 7113	80
546	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
547	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
548	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	79
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550	74 0363	74 0442	74 0521	74 0600	74 0678	74 0757	74 0836	74 0915	74 0994	74 1073	79
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552	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	79
553	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	78
554	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
555	74 4293	74 4371	74 4449	74 4528	74 4606	74 4684	74 4762	74 4840	74 4919	74 4997	78
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
557	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
558	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	78
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561	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659	77
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563	75 0508	75 0586	75 0663	75 0740	0817	0894	0971	1048	1125	1202	77
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566	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	77
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568	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
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571	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	76
572	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	76
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574	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	76
575	75 9668	75 9743	75 9819	75 9894	75 9970	76 0045	76 0121	76 0196	76 0272	76 0347	75
576	76 0422	76 0498	76 0573	76 0649	76 0724	0799	0875	0950	1025	1101	75
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578	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	75
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584	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	74
585	76 7156	76 7230	76 7304	76 7379	76 7453	76 7527	76 7601	76 7675	76 7749	76 7823	74
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590	77 0852	77 0926	77 0999	77 1073	77 1146	77 1220	77 1293	77 1367	77 1440	77 1514	74
591	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	73
592	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	73
593	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	73
594	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
595	77 4517	77 4590	77 4663	77 4736	77 4809	77 4882	77 4955	77 5028	77 5100	77 5173	73
596	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
597	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	73
598	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	73
599	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	72
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604	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	72
605	78 1755	78 1827	78 1899	78 1971	78 2042	78 2114	78 2186	78 2258	78 2329	78 2401	72
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607	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	71
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610	78 5330	78 5401	78 5472	78 5543	78 5615	78 5686	78 5757	78 5828	78 5899	78 5970	71
611	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
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613	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	71
614	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	71
615	78 8875	78 8946	78 9016	78 9087	78 9157	78 9228	78 9299	78 9369	78 9440	78 9510	71
616	9581	9651	9722	9792	9863	9933	79 0004	79 0074	79 0144	79 0215	70
617	79 0285	79 0356	79 0426	79 0496	79 0567	79 0637	0707	0778	0848	0918	70
618	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	70
619	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	70
620	79 2392	79 2462	79 2532	79 2602	79 2672	79 2742	79 2812	79 2882	79 2952	79 3022	70
621	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721	70
622	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	70
623	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	70
624	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	70
625	79 5880	79 5949	79 6019	79 6088	79 6158	79 6227	79 6297	79 6366	79 6436	79 6505	69
626	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
627	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
628	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	69
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630	79 9341	79 9409	79 9478	79 9547	79 9616	79 9685	79 9754	79 9823	79 9892	79 9961	69
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632	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	69
633	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	69
634	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	68
635	80 2774	80 2842	80 2910	80 2979	80 3047	80 3116	80 3184	80 3252	80 3321	80 3389	68
636	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	68
637	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
638	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
639	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	68
640	80 6180	80 6248	80 6316	80 6384	80 6451	80 6519	80 6587	80 6655	80 6723	80 6790	68
641	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	68
642	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	68
643	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	67
644	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	67
645	80 9560	80 9627	80 9694	80 9762	80 9829	80 9896	80 9964	81 0031	81 0098	81 0165	67
646	81 0233	81 0300	81 0367	81 0434	81 0501	81 0569	81 0636	0703	0770	0837	67
647	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
648	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	67
649	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	67
650	81 2913	81 2980	81 3047	81 3114	81 3181	81 3247	81 3314	81 3381	81 3448	81 3514	67
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652	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	67
653	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	66
654	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	66
655	81 6241	81 6308	81 6374	81 6440	81 6506	81 6573	81 6639	81 6705	81 6771	81 6838	66
656	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	66
657	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	66
658	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	66
659	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	66
660	81 9544	81 9610	81 9676	81 9741	81 9807	81 9873	81 9939	82 0004	82 0070	82 0136	66
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662	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448	66
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665	82 2822	82 2887	82 2952	82 3018	82 3083	82 3148	82 3213	82 3279	82 3344	82 3409	65
666	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	65
667	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	65
668	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	65
669	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	65
670	82 6075	82 6140	82 6204	82 6269	82 6334	82 6399	82 6464	82 6528	82 6593	82 6658	65
671	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	65
672	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	65
673	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	64
674	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	64
675	82 9304	82 9368	82 9432	82 9497	82 9561	82 9625	82 9690	82 9754	82 9818	82 9882	64
676	9947	83 0011	83 0075	83 0139	83 0204	83 0268	83 0332	83 0396	83 0460	83 0525	64
677	83 0589	0653	0717	0781	0845	0909	0973	1037	1102	1166	64
678	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	64
679	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	64
680	83 2509	83 2573	83 2637	83 2700	83 2764	83 2828	83 2892	83 2956	83 3020	83 3083	64
681	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721	64
682	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357	64
683	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	63
684	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	63
685	83 5691	83 5754	83 5817	83 5881	83 5944	83 6007	83 6071	83 6134	83 6197	83 6261	63
686	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	63
687	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	62
688	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	62
689	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
690	83 8849	83 8912	83 8975	83 9038	83 9101	83 9164	83 9227	83 9289	83 9352	83 9415	63
691	9478	9541	9604	9667	9729	9792	9855	9918	9981	84 0043	63
692	84 0106	84 0169	84 0232	84 0294	84 0357	84 0420	84 0482	84 0545	84 0608	0671	63
693	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297	63
694	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	62
695	84 1985	84 2047	84 2110	84 2172	84 2235	84 2297	84 2360	84 2422	84 2484	84 2547	62
696	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	62
697	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	62
698	3853	3918	3980	4042	4104	4166	4229	4291	4353	4415	62
699	4477	4539	4601	4664	4726	4788	4850	4912	4974	5036	62
700	84 5098	84 5160	84 5222	84 5284	84 5346	84 5408	84 5470	84 5532	84 5594	84 5656	62
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700	84 5098	84 5160	84 5222	84 5284	84 5346	84 5408	84 5470	84 5532	84 5594	84 5656	62
701	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	62
702	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	62
703	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	62
704	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	62
705	84 8189	84 8251	84 8312	84 8374	84 8435	84 8497	84 8559	84 8620	84 8682	84 8743	62
706	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
707	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	61
708	85 0033	85 0095	85 0156	85 0217	85 0279	85 0340	85 0401	85 0462	85 0524	85 0585	61
709	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	61
710	85 1258	85 1320	85 1381	85 1442	85 1503	85 1564	85 1625	85 1686	85 1747	85 1809	61
711	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419	61
712	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029	61
713	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	61
714	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	61
715	85 4306	85 4367	85 4428	85 4488	85 4549	85 4610	85 4670	85 4731	85 4792	85 4852	61
716	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	61
717	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	61
718	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	60
719	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	61
720	85 7332	85 7393	85 7453	85 7513	85 7574	85 7634	85 7694	85 7755	85 7815	85 7875	60
721	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477	60
722	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078	60
723	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679	60
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725	86 0338	86 0398	86 0458	86 0518	86 0578	86 0637	86 0697	86 0757	86 0817	86 0877	60
726	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	60
727	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	60
728	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	60
729	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	60
730	86 3323	86 3382	86 3442	86 3501	86 3561	86 3620	86 3680	86 3739	86 3799	86 3858	59
731	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	59
732	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	59
733	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	59
734	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	59
735	86 6287	86 6346	86 6405	86 6465	86 6524	86 6583	86 6642	86 6701	86 6760	86 6819	59
736	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	59
737	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	59
738	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	59
739	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173	59
740	86 9232	86 9290	86 9349	86 9408	86 9466	86 9525	86 9584	86 9642	86 9701	86 9760	59
741	9818	9877	9935	9994	87 0053	87 0111	87 0170	87 0228	87 0287	87 0345	59
742	87 0404	87 0462	87 0521	87 0579	0638	0696	0755	0813	0872	0930	58
743	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	58
744	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	58
745	87 2156	87 2215	87 2273	87 2331	87 2389	87 2448	87 2506	87 2564	87 2622	87 2681	58
746	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	58
747	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	58
748	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	58
749	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	58
750	87 5061	87 5119	87 5177	87 5235	87 5293	87 5351	87 5409	87 5466	87 5524	87 5582	58
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751	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	58
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753	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	58
754	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	58
755	87 7947	87 8004	87 8062	87 8119	87 8177	87 8234	87 8292	87 8349	87 8407	87 8464	57
756	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	57
757	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	57
758	9669	9726	9784	9841	9898	9956	88 0013	88 0070	88 0127	88 0185	57
759	88 0242	88 0299	88 0356	88 0413	88 0471	88 0528	0585	0642	0699	0756	57
760	88 0814	88 0871	88 0928	88 0985	88 1042	88 1099	88 1156	88 1213	88 1271	88 1328	57
761	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898	57
762	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	57
763	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	57
764	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	57
765	88 3661	88 3718	88 3775	88 3832	88 3888	88 3945	88 4002	88 4059	88 4116	88 4172	57
766	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	57
767	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	57
768	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	57
769	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	56
770	88 6491	88 6547	88 6604	88 6660	88 6716	88 6773	88 6829	88 6885	88 6942	88 6998	56
771	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	56
772	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	56
773	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	56
774	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	56
775	88 9302	88 9358	88 9414	88 9470	88 9526	88 9582	88 9638	88 9694	88 9750	88 9806	56
776	9862	9918	9974	89 0030	89 0086	89 0141	89 0197	89 0253	89 0309	89 0365	56
777	89 0421	89 0477	89 0533	0589	0645	0700	0756	0812	0868	0924	56
778	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	56
779	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	56
780	89 2095	89 2150	89 2206	89 2262	89 2317	89 2373	89 2429	89 2484	89 2540	89 2595	56
781	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	56
782	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	56
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785	89 4870	89 4925	89 4980	89 5036	89 5091	89 5146	89 5201	89 5257	89 5312	89 5367	55
786	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	55
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788	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	55
789	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
790	89 7627	89 7682	89 7737	89 7792	89 7847	89 7902	89 7957	89 8012	89 8067	89 8122	55
791	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	55
792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	55
793	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	55
794	9821	9875	9930	9985	90 0039	90 0094	90 0149	90 0203	90 0258	90 0312	55
795	90 0367	90 0422	90 0476	90 0531	90 0586	90 0640	90 0695	90 0749	90 0804	90 0859	55
796	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	55
797	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	54
798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
799	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	54
800	90 3090	90 3144	90 3199	90 3253	90 3307	90 3361	90 3416	90 3470	90 3524	90 3578	54
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801	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	54
802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	54
803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	54
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
805	90 5796	90 5850	90 5904	90 5958	90 6012	90 6066	90 6119	90 6173	90 6227	90 6281	54
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	54
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	54
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	54
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
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811	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	53
812	9556	9610	9663	9716	9770	9823	9877	9930	9984	91 0037	53
813	91 0091	91 0144	91 0197	91 0251	91 0304	91 0358	91 0411	91 0464	91 0518	0571	53
814	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	53
815	91 1158	91 1211	91 1264	91 1317	91 1371	91 1424	91 1477	91 1530	91 1584	91 1637	53
816	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
817	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
818	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
819	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761	53
820	91 3814	91 3867	91 3920	91 3973	91 4026	91 4079	91 4132	91 4184	91 4237	91 4290	53
821	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819	53
822	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	53
823	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	53
824	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	53
825	91 6454	91 6507	91 6559	91 6612	91 6664	91 6717	91 6770	91 6822	91 6875	91 6927	53
826	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	53
827	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	52
828	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	52
829	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	52
830	91 9078	91 9130	91 9183	91 9235	91 9287	91 9340	91 9392	91 9444	91 9496	91 9549	52
831	9601	9653	9706	9758	9810	9862	9914	9967	92 0019	92 0071	52
832	92 0123	92 0176	92 0228	92 0280	92 0332	92 0384	92 0436	92 0489	0541	0593	52
833	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	52
834	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
835	92 1686	92 1738	92 1790	92 1842	92 1894	92 1946	92 1998	92 2050	92 2102	92 2154	52
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837	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	52
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840	92 4279	92 4331	92 4383	92 4434	92 4486	92 4538	92 4589	92 4641	92 4693	92 4744	52
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	52
842	5312	5364	5415	5467	5519	5570	5621	5673	5725	5776	52
843	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
844	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	51
845	92 6857	92 6908	92 6959	92 7011	92 7062	92 7114	92 7165	92 7216	92 7268	92 7319	51
846	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	51
847	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	51
848	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	51
849	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	51
850	92 9419	92 9470	92 9521	92 9572	92 9623	92 9674	92 9725	92 9776	92 9827	92 9879	51
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851	9930	9981	93 0032	93 0083	93 0134	93 0185	93 0236	93 0287	93 0338	93 0389	51
852	93 0440	93 0491	0542	0592	0643	0694	0745	0796	0847	0898	51
853	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	51
854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	51
855	93 1966	93 2017	93 2068	93 2118	93 2169	93 2220	93 2271	93 2322	93 2372	93 2423	51
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	51
857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	51
858	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	51
859	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	51
860	93 4498	93 4549	93 4599	93 4650	93 4700	93 4751	93 4801	93 4852	93 4902	93 4953	50
861	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	50
862	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	50
863	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	50
864	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	50
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866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	50
867	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	50
868	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	50
869	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	50
870	93 9519	93 9569	93 9619	93 9669	93 9719	93 9769	93 9819	93 9869	93 9918	93 9968	50
871	94 0018	94 0068	94 0118	94 0168	94 0218	94 0267	94 0317	94 0367	94 0417	94 0467	50
872	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964	50
873	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	50
874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	50
875	94 2008	94 2058	94 2107	94 2157	94 2207	94 2256	94 2306	94 2355	94 2405	94 2455	50
876	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	50
877	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	49
878	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	49
879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	49
880	94 4483	94 4532	94 4581	94 4631	94 4680	94 4729	94 4779	94 4828	94 4877	94 4927	49
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	49
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	49
883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	49
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	49
885	94 6943	94 6992	94 7041	94 7090	94 7139	94 7189	94 7238	94 7287	94 7336	94 7385	49
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	49
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	49
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	49
890	94 9390	94 9439	94 9488	94 9536	94 9585	94 9634	94 9683	94 9731	94 9780	94 9829	49
891	9878	9926	9975	0024	0073	0121	0170	0219	0267	0316	49
892	95 0365	95 0414	95 0462	0511	0560	0608	0657	0706	0754	0803	49
893	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	49
894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	49
895	95 1823	95 1872	95 1920	95 1969	95 2017	95 2066	95 2114	95 2163	95 2211	95 2260	48
896	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
897	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	48
898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	48
899	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	48
900	95 4243	95 4291	95 4339	95 4387	95 4435	95 4484	95 4532	95 4580	95 4628	95 4677	48
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
900	95 4243	95 4291	95 4339	95 4387	95 4435	95 4484	95 4532	95 4580	95 4628	95 4677	48
901	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	48
902	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640	48
903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	48
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
905	95 6649	95 6697	95 6745	95 6793	95 6840	95 6888	95 6936	95 6984	95 7032	95 7080	48
906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	48
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	48
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	48
909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
910	95 9041	95 9089	95 9137	95 9185	95 9232	95 9280	95 9328	95 9375	95 9423	95 9471	48
911	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	48
912	9995	96 0042	96 0090	96 0138	96 0185	96 0233	96 0280	96 0328	96 0376	96 0423	48
913	96 0471	0518	0566	0613	0661	0709	0756	0804	0851	0899	48
914	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	48
915	96 1421	96 1469	96 1516	96 1563	96 1611	96 1658	96 1706	96 1753	96 1801	96 1848	48
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	47
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	47
919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	47
920	96 3788	96 3835	96 3882	96 3929	96 3977	96 4024	96 4071	96 4118	96 4165	96 4212	47
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	47
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
925	96 6142	96 6189	96 6236	96 6283	96 6329	96 6376	96 6423	96 6470	96 6517	96 6564	47
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
930	96 8483	96 8530	96 8576	96 8623	96 8670	96 8716	96 8763	96 8810	96 8856	96 8903	47
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	47
932	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	47
933	9882	9928	9975	97 0021	97 0068	97 0114	97 0161	97 0207	97 0254	97 0300	47
934	97 0347	97 0393	97 0440	0486	0533	0579	0626	0672	0719	0765	46
935	97 0812	97 0858	97 0904	97 0951	97 0997	97 1044	97 1090	97 1137	97 1183	97 1229	46
936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	46
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	46
940	97 3128	97 3174	97 3220	97 3266	97 3313	97 3359	97 3405	97 3451	97 3497	97 3543	46
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	46
942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
945	97 5432	97 5478	97 5524	97 5570	97 5616	97 5662	97 5707	97 5753	97 5799	97 5845	46
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
950	97 7724	97 7769	97 7815	97 7861	97 7906	97 7952	97 7998	97 8043	97 8089	97 8135	46
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
950	97 7724	97 7769	97 7815	97 7861	97 7906	97 7952	97 7998	97 8043	97 8089	97 8135	46
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
955	98 0003	98 0049	98 0094	98 0140	98 0185	98 0231	98 0276	98 0322	98 0367	98 0412	45
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
960	98 2271	98 2316	98 2362	98 2407	98 2452	98 2497	98 2543	98 2588	98 2633	98 2678	45
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
965	98 4527	98 4572	98 4617	98 4662	98 4707	98 4752	98 4797	98 4842	98 4887	98 4932	45
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	45
967	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
970	98 6772	98 6817	98 6861	98 6906	98 6951	98 6996	98 7040	98 7085	98 7130	98 7175	45
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	45
975	98 9005	98 9049	98 9094	98 9138	98 9183	98 9227	98 9272	98 9316	98 9361	98 9405	45
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
977	9895	9939	9983	99 0028	99 0072	99 0117	99 0161	99 0206	99 0250	99 0294	44
978	99 0339	99 0383	99 0428	0472	0516	0561	0605	0650	0694	0738	44
979	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	44
980	99 1226	99 1270	99 1315	99 1359	99 1403	99 1448	99 1492	99 1536	99 1580	99 1625	44
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
985	99 3436	99 3480	99 3524	99 3568	99 3613	99 3657	99 3701	99 3745	99 3789	99 3833	44
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
988	4757	4801	4845	4889	4933	4977	5021	5065	5109	5152	44
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	99 5635	99 5679	99 5723	99 5767	99 5811	99 5854	99 5898	99 5942	99 5986	99 6030	44
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
995	99 7823	99 7867	99 7910	99 7954	99 7998	99 8041	99 8085	99 8129	99 8172	99 8216	44
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	44
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43
N.	0	1	2	3	4	5	6	7	8	9	D.



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TRIGONOMETRY

DEFINITIONS.

1. **Geometry** shows how to *construct* a triangle and how to find the unknown parts by actual measurement. **Trigonometry** shows how to *compute* the unknown parts from the numerical values of the known parts.

In any triangle there are six parts: three sides and three angles. These six parts are so related that any three, provided one of them is a side, determine the shape and size of the triangle.

2. *Trigonometry*, then, is that branch of mathematics which treats of the numerical computation of *angles* and *triangles*.

3. In deducing rules or formulas to be used in computations, we use what are called **functions** of the angles. This word function is used in mathematics to denote a quantity which depends on some other quantity for its value. Thus the area of a circle depends upon the radius of a circle for its value; hence the area of a circle is a function of the radius.

4. Let $\angle MAN$ be any acute angle. From B , D and F , any points in AN , draw lines perpendicular to AM , forming the right triangles BCA , DEA and FGA . These right triangles are similar, since they are mutually equiangular. The homologous sides are then proportional. Therefore :

$$\frac{CB}{AB} = \frac{ED}{AD} = \frac{GF}{AF} \text{ and } \frac{CB}{AC} = \frac{ED}{AE} = \frac{GF}{AG}, \text{ etc.}$$

These ratios, which depend upon the magnitude of angle A

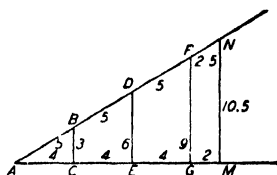


Fig. 1.

for their values, are functions of A . There are six such ratios and they are defined and named as follows :

$$\frac{CB}{AB} = \text{sine of } A \text{ (written } \sin A \text{)}.$$

$$\frac{AC}{AB} = \text{cosine of } A \text{ (written } \cos A \text{)}.$$

$$\frac{CB}{AC} = \text{tangent of } A \text{ (written } \tan A \text{)}.$$

$$\frac{AC}{CB} = \text{cotangent of } A \text{ (written } \cot A \text{)}.$$

$$\frac{AB}{AC} = \text{secant of } A \text{ (written } \sec A \text{)}.$$

$$\frac{AB}{CB} = \text{cosecant of } A \text{ (written } \csc A \text{)}.$$

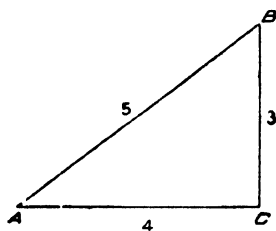


Fig. 2.

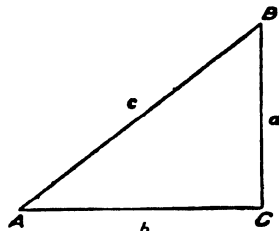


Fig. 3.

Example.—Take the right triangle ACB , Fig. 2, with the sides 3, 4, 5. Then from our definitions :

$$\sin A = \frac{3}{5} \qquad \cot A = \frac{4}{3}$$

$$\cos A = \frac{4}{5} \qquad \sec A = \frac{5}{4}$$

$$\tan A = \frac{3}{4} \qquad \csc A = \frac{5}{3}$$

5. When in Art. 13 to Art. 17 we take up the solution of right triangles, we shall use a triangle lettered as in Fig. 3. A and B are the acute angles, C the right angle, and the opposite sides are represented by the corresponding small letters. That is, the side opposite angle A is a , that opposite B is b , etc.

Then,

$$\left. \begin{aligned}
 \sin A &= \frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}} & \text{Rule 1} \\
 \cos A &= \frac{b}{c} = \frac{\text{adjacent side}}{\text{hypotenuse}} & \text{" 2} \\
 \tan A &= \frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}} & \text{" 3} \\
 \cot A &= \frac{b}{a} = \frac{\text{adjacent side}}{\text{opposite side}} & \text{" 4} \\
 \sec A &= \frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent side}} & \text{" 5} \\
 \csc A &= \frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite side}} & \text{" 6}
 \end{aligned} \right\} \quad [1]$$

The six ratios are called the *Trigonometric Functions* of angle A.

6. The following definitions must be committed to memory :

The *sine* of an angle is the ratio of the opposite side to the hypotenuse.

The *cosine* of an angle is the ratio of the adjacent side to the hypotenuse.

The *tangent* of an angle is the ratio of the opposite side to the adjacent side.

The *cotangent* of an angle is the ratio of the adjacent side to the opposite side.

The *secant* of an angle is the ratio of the hypotenuse to the adjacent side.

The *cosecant* of an angle is the ratio of the hypotenuse to the opposite side.

7. If we apply these definitions to angle B we have the following :

$$\begin{aligned}
 \sin B &= \frac{b}{c} & \cot B &= \frac{a}{b} \\
 \cos B &= \frac{a}{c} & \sec B &= \frac{c}{a} \\
 \tan B &= \frac{b}{a} & \csc B &= \frac{c}{b}
 \end{aligned}$$

By inspection of the functions of A and B (which are complementary angles) we see that :

$$\left. \begin{aligned}
 \sin A &= \cos B & \cos A &= \sin B \\
 \tan A &= \cot B & \cot A &= \tan B \\
 \sec A &= \csc B & \csc A &= \sec B
 \end{aligned} \right\} \quad [2]$$

EXAMPLES FOR PRACTICE.

Calculate the value of the six functions of angle A in the right triangle ACB , whose sides a, b, c are respectively:

- (1) 40, 9, 41.
- (2) 8, 15, 17.
- (3) 5, 12, 13.
- (4) 196, 315, 371.

8. Given one function of an angle to find the other functions. Suppose we have given $\sin A = \frac{1}{2}$, and the other five functions are required. We construct a right triangle ACB (Fig. 4) calling the opposite side 1 and the hypotenuse 2. We can do this because

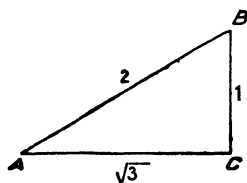


Fig. 4.

$$\sin A = \frac{\text{opp. side}}{\text{hyp.}} = \frac{1}{2}$$

By Geometry $AC = \sqrt{AB^2 - CB^2} = \sqrt{4 - 1} = \sqrt{3}$

Hence the adjacent side is $\sqrt{3}$.

$$\text{Therefore} \quad \cos A = \frac{\sqrt{3}}{2} \quad \sec A = \frac{2}{\sqrt{3}}$$

$$\tan A = \frac{1}{\sqrt{3}}$$

$$\cot A = \sqrt{3} \quad \csc A = 2.$$

EXAMPLES FOR PRACTICE.

Find the values of the five remaining functions, when

1. $\tan A = \frac{2}{3}$
3. $\sin A = \frac{3}{8}$
2. $\csc A = 4$ (or $\frac{1}{4}$)
4. $\cot A = \frac{4}{7}$

Measurement of Angles. As we learned in "Mensuration," angles are usually expressed in degrees, minutes and seconds. A degree is a ninetieth part of a right angle; a minute is one-sixtieth of a degree, and a second is one-sixtieth of a minute. The symbols used to denote degrees, minutes and seconds are $^{\circ}$ ' ". Thus forty-six degrees fifteen minutes and six seconds is written $46^{\circ} 15' 6''$. In adding two angles, seconds are added to seconds,

minutes to minutes, and degrees to degrees; also, in subtracting two angles, seconds are subtracted from seconds, minutes from minutes, and degrees from degrees.

Example 1.—Add $67^{\circ} 45' 16''$ and $32^{\circ} 28' 52''$.

Solution :

$$\begin{array}{r} 67^{\circ} 45' 16'' \\ 32^{\circ} 28' 52'' \\ \hline 99^{\circ} 73' 68'' \end{array}$$

But $68'' = 1' 8''$ and $73' = 1^{\circ} 13'$.

Therefore $99^{\circ} 73' 68'' = 100^{\circ} 14' 8''$ Ans.

Example 2.—Subtract $35^{\circ} 28' 14''$ from $119^{\circ} 14' 25''$.

Solution :

$$\begin{array}{r} 119^{\circ} 14' 25'' \\ 35^{\circ} 28' 14'' \\ \hline \end{array}$$

Since $28'$ cannot be taken from $14'$ we borrow $1^{\circ} (= 60')$ from 119° and write the angles

$$\begin{array}{r} 118^{\circ} 74' 25'' \\ 35^{\circ} 28' 14'' \\ \hline 83^{\circ} 46' 11'' \end{array} \text{ Ans.}$$

Example 3.—In a right triangle one acute angle is $16^{\circ} 25' 13''$, find the other acute angle.

We know that the sum of the acute angles of a right triangle is equal to one right angle. We write 90° as $89^{\circ} 59' 60''$, then subtracting the given angle,

$$\begin{array}{r} 89^{\circ} 59' 60'' \\ 16^{\circ} 25' 13'' \\ \hline 73^{\circ} 34' 47'' \end{array} \text{ Ans.}$$

EXAMPLES FOR PRACTICE.

1. Add $176^{\circ} 59' 24''$ and $3^{\circ} 12' 56''$. Ans. $180^{\circ} 12' 20''$.
2. Add $56^{\circ} 28' 32''$ and $39^{\circ} 43' 17''$. Ans. $96^{\circ} 11' 49''$.
3. From $84^{\circ} 19' 27''$ take $44^{\circ} 20' 34''$. Ans. $39^{\circ} 58' 53''$.

10. **The Functions of 45° .** In order to find the functions of 45° take a right isosceles triangle A C B (Fig. 5), each side about the right angle being equal to unity. Then

$$AB = \sqrt{AC^2 + BC^2} = \sqrt{1+1} = \sqrt{2}. \quad \angle A + \angle B = 90^\circ$$

and as $A = B$ each equals 45° . Therefore, by definition :

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \cot 45^\circ = 1.$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \sec 45^\circ = \sqrt{2}.$$

$$\tan 45^\circ = 1 \quad \csc 45^\circ = \sqrt{2}.$$

In the above, the radical sign is removed from the denominator by multiplying both numerator and denominator by the denominator.

$$\text{Then,} \quad \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{2}}{2}.$$

This process is called rationalizing the denominator.

11. Functions of 30° and 60° . To find the functions of 30° and 60° , take an equilateral triangle ABD (Fig. 6). Assume each side equal to 2. Draw AC perpendicular to BD . Then by geometry, $BC = CD$ and angle $BAC = \text{angle } CAD$.

$$\text{Also} \quad AC = \sqrt{AB^2 - BC^2} = \sqrt{4 - 1} = \sqrt{3}.$$

But if ABD is equilateral, the triangle is equiangular, and each angle is 60° . Hence $\angle ABC = 60^\circ$ and $\angle BAC$ or one-half of $\angle BAD = 30^\circ$.

Applying our definitions to the right triangle ACB :

$$\sin CAB = \sin 30^\circ = \frac{1}{2} \quad \cot CAB = \cot 30^\circ = \sqrt{3}$$

$$\cos CAB = \cos 30^\circ = \frac{\sqrt{3}}{2} \quad \sec CAB = \sec 30^\circ = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\tan CAB = \tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \quad \csc CAB = \csc 30^\circ = 2.$$

$$\sin CBA = \sin 60^\circ = \frac{\sqrt{3}}{2} \quad \cot CBA = \cot 60^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cos CBA = \cos 60^\circ = \frac{1}{2} \quad \sec CBA = \sec 60^\circ = 2$$

$$\tan CBA = \tan 60^\circ = \sqrt{3} \quad \csc CBA = \csc 60^\circ = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

FUNDAMENTAL THEOREMS.

12. Since $\sin A = \frac{a}{c}$ and $\csc A = \frac{c}{a}$: (Fig. 3)

Since $\cos A = \frac{b}{c}$ and $\sec A = \frac{c}{b}$

Since $\tan A = \frac{a}{b}$ and $\cot A = \frac{b}{a}$

It is evident that

$$\left. \begin{aligned} \sin A &= \frac{1}{\csc A}; & \cos A &= \frac{1}{\sec A}; & \tan A &= \frac{1}{\cot A} \\ \csc A &= \frac{1}{\sin A}; & \sec A &= \frac{1}{\cos A}; & \cot A &= \frac{1}{\tan A} \end{aligned} \right\} [3]$$

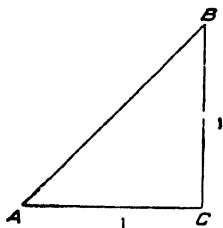


Fig. 5.

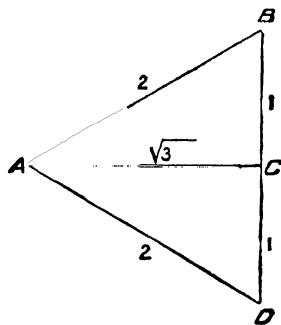


Fig. 6.

13. In the right triangle ABC, Fig. 3,

By Geometry $BC^2 + AC^2 = AB^2$

Or $a^2 + b^2 = c^2$

Dividing by c^2 $\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}$

That is $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$

Whence by definition $(\sin A)^2 + (\cos A)^2 = 1$

This is usually written,

$$\sin^2 A + \cos^2 A = 1 \quad [4]$$

(It is read, sine squared A plus cosine squared A equals one.)

14. By our definition $\tan A = \frac{a}{b}$

Divide numerator and denominator by c

$$\text{Then} \quad \tan A = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{\sin A}{\cos A} \quad [5]$$

$$\text{Similarly,} \quad \cot A = \frac{\frac{c}{b}}{\frac{c}{a}} = \frac{\cos A}{\sin A} \quad [6]$$

15. In triangle A B C, Fig. 3,

$$\text{By Geometry} \quad \overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$$

$$\text{Or} \quad c^2 = b^2 + a^2$$

$$\text{Divide by } b^2, \quad \frac{c^2}{b^2} = 1 + \frac{a^2}{b^2}$$

$$\text{That is} \quad \left(\frac{c}{b}\right)^2 = 1 + \left(\frac{a}{b}\right)^2$$

$$16. \text{ By definition} \quad \sec^2 A = 1 + \tan^2 A \quad [7]$$

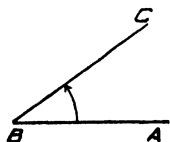
$$\text{Similarly divide} \quad c^2 = b^2 + a^2 \text{ by } a^2$$

$$\frac{c^2}{a^2} = \frac{b^2}{a^2} + 1$$

$$\text{Whence by definition} \quad \csc^2 A = \cot^2 A + 1 \quad [8].$$

FUNCTIONS OF ANY ANGLE.

17. In Art. 4, definitions for the functions of *acute* angles only were given. Before we can define the functions of *any* angle we must consider angles themselves.

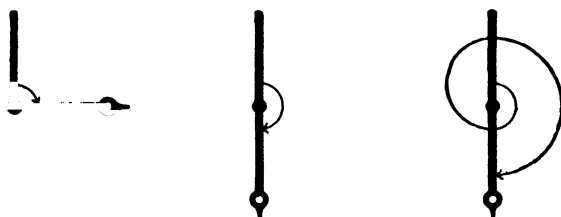


In Trigonometry an angle is considered as generated by revolving a line from some initial position (called initial line) to some final position (called terminal line). Thus the angle A B C may be generated by revolving a line from the position B A to the position B C.

If the minute hand of a clock makes $\frac{1}{4}$ of a revolution, it turns through one right angle; if it makes one-half a revolution,

it turns through two right angles; if one and one-half revolutions, it turns through six right angles. Thus :

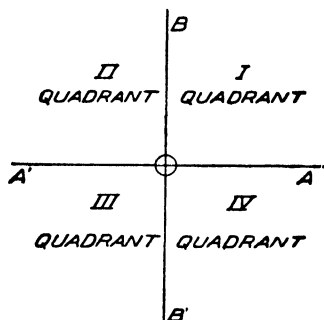
By revolving a sufficient number of times it is evident that an angle of any magnitude may be formed.



18. Positive Angles. If the direction of revolution is *opposite* to that of the hands of a clock, the angle is said to be *positive*.

Negative Angles. If the direction of revolution is the same as that of the hands of a clock, the angle is said to be *negative*. The angles in Art. 17 are all negative.

Example.—Through how many right angles does the hour hand of a clock move in $8\frac{1}{2}$ hours? The minute hand?

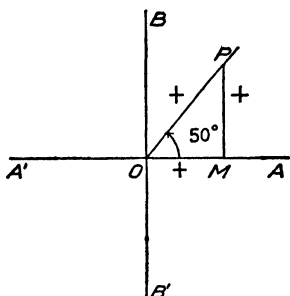


Example.—If the fly wheel of an engine makes 60 revolutions per minute, through how many right angles does a spoke of the fly wheel revolve in one second?

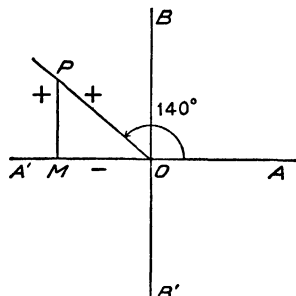
19. Let BB' be drawn perpendicular to AA' , intersecting it at O . The plane of the page is then divided into four parts

called quadrants. $A O B$ is called the first quadrant, $B O A'$ is called the second quadrant, $A' O B'$ is called the third quadrant, and $B' O A$ is called the fourth quadrant.

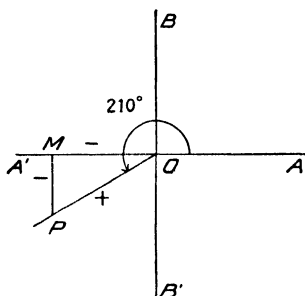
FIRST QUADRANT



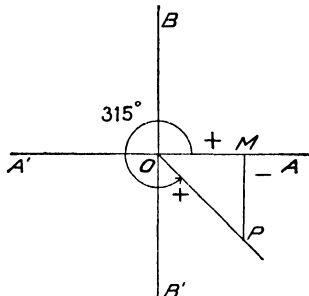
SECOND QUADRANT



THIRD QUADRANT



FOURTH QUADRANT



An angle is said to lie in the quadrant in which its terminal line lies.

20. Construct an angle $A O P$ in each of the four quadrants. Draw $P M$ perpendicular to $A A'$, meeting it at M , forming the triangle $M O P$. The sides of this triangle determine the functions of the angle. If the perpendicular $M P$ is drawn above $A A'$, it is considered *positive*; if below $A A'$, it is considered *negative*. If the base $O M$ is drawn to the right of $B B'$, it is considered positive; if drawn to the left of $B B'$, it is considered negative. $O P$ is always positive. Hence, some of the functions of angles in the second, third and fourth quadrants are negative.

21. The trigonometric functions of any angle then are,

$$\sin AOP = \frac{\text{perpendicular}}{\text{hypotenuse}} \quad \left\{ \begin{array}{l} + \text{ in I and II.} \\ - \text{ in III and IV.} \end{array} \right.$$

$$\cos AOP = \frac{\text{base}}{\text{hypotenuse}} \quad \left\{ \begin{array}{l} + \text{ in I and IV.} \\ - \text{ in II and III.} \end{array} \right.$$

$$\tan AOP = \frac{\text{perpendicular}}{\text{base}} \quad \left\{ \begin{array}{l} + \text{ in I and III.} \\ - \text{ in II and IV.} \end{array} \right.$$

$$\cot AOP = \frac{\text{base}}{\text{perpendicular}} \quad \left\{ \begin{array}{l} + \text{ in I and III.} \\ - \text{ in II and IV.} \end{array} \right.$$

$$\sec AOP = \frac{\text{hypotenuse}}{\text{base}} \quad \left\{ \begin{array}{l} + \text{ in I and IV.} \\ - \text{ in II and III.} \end{array} \right.$$

$$\csc AOP = \frac{\text{hypotenuse}}{\text{perpendicular}} \quad \left\{ \begin{array}{l} + \text{ in I and II.} \\ - \text{ in III and IV.} \end{array} \right.$$

The signs of the functions may be tabulated thus :

Quadrant	I	II	III	IV
Sine and cosecant	+	+	—	—
Tangent and cotangent	+	—	+	—
Secant and cosine	+	—	—	+

EXAMPLES FOR PRACTICE.

1. Construct the following angles: 150° , 225° , 270° , 300° , 400° , -120° , -45° , 840° .
2. The terminal side of 650° lies in what quadrant?
3. The terminal side of 430° lies in what quadrant?

FUNCTIONS OF 0° , 90° , 180° , 270° .

22. Functions of 0° . Let Fig. 7 represent a circle with a radius equal to unity. Let AOP be an angle in the first quadrant.

$$\text{Then } \sin AOP = \frac{MP}{OP} = \frac{MP}{1} = MP$$

$$\cos AOP = \frac{OM}{OP} = \frac{OM}{1} = OM$$

Let OP be revolved towards OA , the angle AOP growing smaller and smaller and approaching zero degrees as OP approaches the position OA . Then MP approaches zero and OM approaches OA .

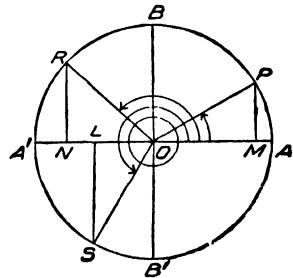


Fig. 7.

$$\sin 0^\circ = \frac{MP}{OP} = \frac{0}{1} = 0; \quad \cos 0^\circ = \frac{OM}{OP} = \frac{1}{1} = 1$$

Therefore

$$\sin 0^\circ = 0$$

and

$$\cos 0^\circ = OA = 1.$$

$$\text{By formula [5] } \tan 0^\circ = \frac{\sin 0^\circ}{\cos 0^\circ} = \frac{0}{1} = 0$$

$$\text{By formula [6] } \cot 0^\circ = \frac{\cos 0^\circ}{\sin 0^\circ} = \frac{1}{0} = \infty$$

$$\text{By formula [3] } \sec 0^\circ = \frac{1}{\cos 0^\circ} = \frac{1}{1} = 1$$

$$\text{By formula [3] } \csc 0^\circ = \frac{1}{\sin 0^\circ} = \frac{1}{0} = \infty.$$

23. Functions of 90° . Let OP (Fig. 7) be revolved until it coincides with OB . Then $\angle AOP$ becomes 90° , $OM = 0$ and $MP = OB$.

$$\text{Therefore } \sin 90^\circ = OB = 1$$

$$\cos 90^\circ = 0$$

$$\tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \frac{1}{0} = \infty$$

$$\cot 90^\circ = \frac{\cos 90^\circ}{\sin 90^\circ} = \frac{0}{1} = 0$$

$$\sec 90^\circ = \frac{1}{\cos 90^\circ} = \frac{1}{0} = \infty$$

$$\csc 90^\circ = \frac{1}{\sin 90^\circ} = \frac{1}{1} = 1.$$

24. Functions of 180° . Let AOR (Fig. 7) be an angle in the second quadrant.

$$\text{Then } \sin AOR = \frac{NR}{OR} = \frac{NR}{1} = NR$$

$$\cos AOR = \frac{ON}{OR} = \frac{ON}{1} = ON$$

Let OR revolve until it coincides with OA' , then $\angle AOR$ becomes $\angle AOA'$ or 180° , $NR = 0$ and $ON = OA' = -1$.

NOTE. ∞ is the symbol used to denote an infinite quantity, that is, a quantity larger than any assignable quantity. Let us consider the series: $\frac{a}{c}, \frac{10a}{c}, \frac{100a}{c}, \frac{1,000a}{c}$. It is evident that if the series is continued, the absolute value of the numerator may be made greater than any assigned positive quantity, and as the denominator remains the same, the absolute value of the fraction may be made greater than any assigned positive quantity. Similarly it may be proved that the value of a fraction becomes very large when its denominator becomes very small.

(See Art. 20.)

Therefore $\sin 180^\circ = 0$

$\cos 180^\circ = O A = -1$

$\tan 180^\circ = \frac{\sin 180^\circ}{\cos 180^\circ} = \frac{0}{-1} = 0$

$\cot 180^\circ = \frac{\cos 180^\circ}{\sin 180^\circ} = \frac{-1}{0} = \infty$

$\sec 180^\circ = \frac{1}{\cos 180^\circ} = \frac{1}{-1} = -1$

$\csc 180^\circ = \frac{1}{\sin 180^\circ} = \frac{1}{0} = \infty$

25. Functions of 270° . Let A O S be an angle in the third quadrant.

Then $\sin A O S = \frac{L S}{O S} = \frac{L S}{1} = L S$

$\cos A O S = \frac{O L}{O S} = \frac{O L}{1} = O L$

Let O S revolve until it coincides with O B', then $\angle A O S$ becomes $\angle A O B'$, or 270° , O L = O, and L S = O B'.

Therefore $\sin 270^\circ = O B' = -1$

$\cos 270^\circ = 0$

$\tan 270^\circ = \frac{\sin 270^\circ}{\cos 270^\circ} = \frac{-1}{0} = \infty$

$\cot 270^\circ = \frac{\cos 270^\circ}{\sin 270^\circ} = \frac{0}{-1} = 0$

$\sec 270^\circ = \frac{1}{\cos 270^\circ} = \frac{1}{0} = \infty$

$\csc 270^\circ = \frac{1}{\sin 270^\circ} = \frac{1}{-1} = -1$

The foregoing results may be expressed in a tabular form as follows :

Angle.	Sin.	Cos.	Tan	Cot.	Sec.	Csc.
0°	0	1	0	∞	1	∞
90°	1	0	∞	0	∞	1
180°	0	-1	0	∞	-1	∞
270°	-1	0	∞	0	∞	-1

RELATIONS BETWEEN FUNCTIONS OF ANGLES.

26. Functions of $(-x)$, $(180^\circ - x)$, $(180^\circ + x)$, $(360^\circ - x)$.

Let $\angle AOP$, in Figs. 8, 9, 10 and 11 represent angle x and $\angle AOP'$ represent $-x$.

If PM $P'M'$ is drawn perpendicular to AA' the right triangles OMP and OMP' have a side and acute angle of one equal respectively to a side and acute angle of the other and are therefore equal.

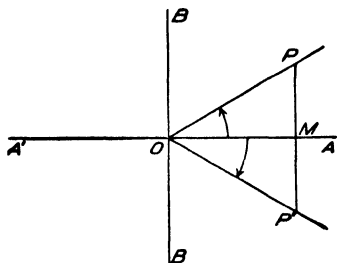


Fig. 8.

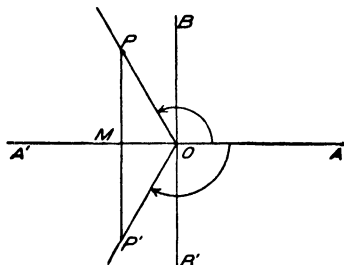


Fig. 9.

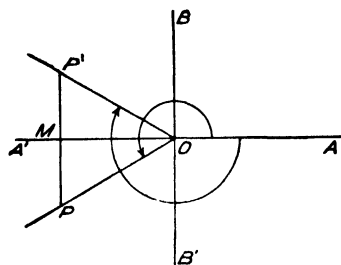


Fig. 10.

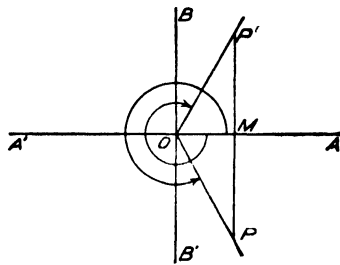


Fig. 11.

Applying the definitions given in Art. 21, noting carefully the signs, we have in Fig. 8, fourth quadrant,

$$\left\{ \begin{array}{lll} \sin(-x) = -\sin x & \tan(-x) = -\tan x & \sec(-x) = +\sec x \\ \cos(-x) = +\cos x & \cot(-x) = -\cot x & \csc(-x) = -\csc x \end{array} \right\} [9]$$

27. If x is an acute angle, $180^\circ - x$ lies in the second quadrant, $180^\circ + x$ in the third quadrant and $360^\circ - x$ in the fourth quadrant.

In Figs. 12, 13 and 14 construct $\angle AOP'$ equal to angle x ; make $OP' = OP$ and draw MP and $M'P'$ perpendicular to AA' .

Then the right triangles $P M O$ and $P' M' O$ are equal, since the hypotenuse and acute angle of one equal the hypotenuse and acute angle of the other.

Applying the definitions given in Art. 21, noting carefully the signs, we have.

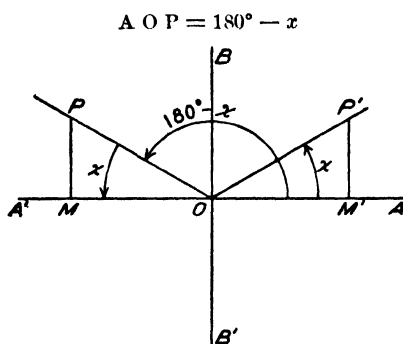


Fig. 12.

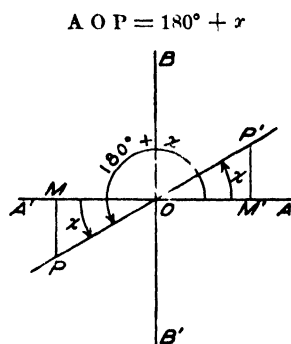


Fig. 13.

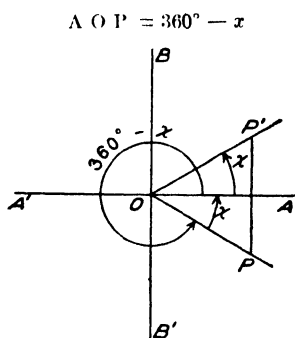


Fig 14.

$$\begin{aligned}
 \text{(Fig 12)} \quad \left\{ \begin{array}{l} \sin (180^\circ - x) = \sin x \\ \cos (180^\circ - x) = -\cos x \\ \tan (180^\circ - x) = -\tan x \end{array} \right. & \quad \left\{ \begin{array}{l} \cot (180^\circ - x) = -\cot x \\ \sec (180^\circ - x) = -\sec x \\ \csc (180^\circ - x) = +\csc x \end{array} \right. \quad [10]
 \end{aligned}$$

Also

$$\begin{aligned}
 \text{(Fig 13)} \quad \left\{ \begin{array}{l} \sin (180^\circ + x) = -\sin x \\ \cos (180^\circ + x) = -\cos x \\ \tan (180^\circ + x) = \tan x \end{array} \right. & \quad \left\{ \begin{array}{l} \cot (180^\circ + x) = \cot x \\ \sec (180^\circ + x) = -\sec x \\ \csc (180^\circ + x) = -\csc x \end{array} \right. \quad [11]
 \end{aligned}$$

Also

$$\text{(Fig. 14)} \left\{ \begin{array}{ll} \sin(360^\circ - x) = -\sin x & \cot(360^\circ - x) = -\cot x \\ \cos(360^\circ - x) = +\cos x & \sec(360^\circ - x) = \sec x \\ \tan(360^\circ - x) = -\tan x & \csc(360^\circ - x) = -\csc x \end{array} \right\} [12]$$

28. To find a function of an angle greater than 360° , subtract a multiple of 360° from the angle, the function remaining the same.

$$\sin 890^\circ = \sin(890^\circ - 720^\circ) = \sin 170^\circ$$

$$= \sin(180^\circ - 10^\circ) = \sin 10^\circ$$

$$\tan 480^\circ = \tan(480^\circ - 360^\circ) = \tan 120^\circ$$

$$= \tan(180^\circ - 60^\circ) = -\tan 60^\circ = -\sqrt{3}$$

$$\csc(-120^\circ) = -\csc 120^\circ = -\csc(180^\circ - 60^\circ) = -\frac{2}{\sqrt{3}}$$

EXAMPLES FOR PRACTICE.

1. $\cos 495^\circ$

4. $\sin 150^\circ$

2. $\tan 330^\circ$

5. $\cos(-60^\circ)$

3. $\sec 225^\circ$

29. Several formulas for the functions of one angle have been derived. We shall now prove four fundamental formulas for functions of two angles and from them derive others.

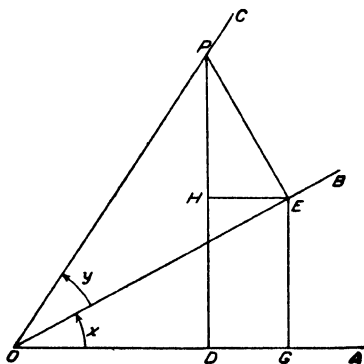
These four formulas are:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \quad [13]$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y \quad [14]$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y \quad [15]$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y \quad [16]$$



angle $\angle HPE$ is equal to angle $\angle AOB$ or x .

30. Let $\angle AOB = x$ and $\angle BOC = y$, then $\angle AOC = (x + y)$. From P , any point in OC , draw PD and PE perpendicular to OA and OB respectively. From E draw EG and EH perpendicular to OA and PD respectively. Since PE is perpendicular to OB , and PH is perpendicular to OA ,

$$\text{Now} \quad \sin (x+y) = \frac{DP}{OP} = \frac{DH+HP}{OP} = \frac{GE}{OP} + \frac{HP}{OP}$$

$$\text{But} \quad \frac{GE}{OP} = \frac{GE}{OE} \times \frac{OE}{OP} = \sin x \cos y$$

$$\text{And} \quad \frac{HP}{OP} = \frac{HP}{EP} \times \frac{EP}{OP} = \cos x \sin y$$

$$\text{Therefore} \quad \sin (x+y) = \sin x \cos y + \cos x \sin y \quad [13]$$

$$\text{Similarly} \quad \cos (x+y) = \frac{OD}{OP} = \frac{OG-DG}{OP} = \frac{OG}{OP} - \frac{HE}{OP}$$

$$\text{But} \quad \frac{OG}{OP} = \frac{OG}{OE} \times \frac{OE}{OP} = \cos x \cos y$$

$$\text{And} \quad \frac{HE}{OP} = \frac{HE}{EP} \times \frac{EP}{OP} = \sin x \sin y$$

$$\text{Therefore} \quad \cos (x+y) = \cos x \cos y - \sin x \sin y \quad [14]$$

31. We have merely proved that these formulas hold true when x and y , and $x+y$ are acute. It can be easily proved by means of diagrams and equations that they are also true for all values of x and y , but the proof will not be taken up here.

32. Substituting $-y$ for y in formula [13], it becomes

$$\sin [x + (-y)] = \sin x \cos (-y) + \cos x \sin (-y)$$

$$\text{But} \quad \cos (-y) = \cos y \text{ and } \sin (-y) = -\sin y \text{ by [9]}$$

$$\text{Therefore} \quad \sin (x-y) = \sin x \cos y - \cos x \sin y \quad [15]$$

Also substituting $-y$ for y in formula [14], it becomes

$$\begin{aligned} \cos [x + (-y)] &= \cos x \cos (-y) - \sin x \sin (-y) \\ &= \cos x \cos y + \sin x \sin y \quad [16] \end{aligned}$$

$$33. \quad \tan (x+y) = \frac{\sin (x+y)}{\cos (x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$

Dividing both numerator and denominator by $\cos x \cos y$, this becomes

$$\tan (x+y) = \frac{\frac{\sin x \cos y}{\cos x \cos y} + \frac{\cos x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y} - \frac{\sin x \sin y}{\cos x \cos y}}$$

34. Cancelling the equal quantities and introducing the formulas $\tan x = \frac{\sin x}{\cos x}$ and $\tan y = \frac{\sin y}{\cos y}$, we finally have

$$\tan (x+y)=\frac{\tan x+\tan y}{1-\tan x \tan y} \quad [17]$$

Similarly

$$\tan (x-y)=\frac{\tan x-\tan y}{1+\tan x \tan y} \quad [18]$$

35. In formula [13] substituting x for y ,

$$\begin{aligned} \sin (x+x) &= \sin x \cos x + \cos x \sin x \\ \text{or} \quad \sin 2 x &= 2 \sin x \cos x \end{aligned} \quad [19]$$

36. In formula [14] substituting x for y ,

$$\begin{aligned} \cos (x+x) &= \cos x \cos x - \sin x \sin x \\ \text{or} \quad \cos 2 x &= \cos^2 x - \sin^2 x \end{aligned} \quad [20]$$

37. If now we substitute for $\sin^2 x$ its equivalent, $1 - \cos^2 x$ (formula [4]), we have,

$$\begin{aligned} \cos 2 x &= \cos^2 x - (1 - \cos^2 x) \\ &= 2 \cos^2 x - 1 \end{aligned} \quad [22]$$

38. If in [20] we substitute for $\cos^2 x$ its equivalent, $1 - \sin^2 x$, we have

$$\begin{aligned} \cos 2 x &= 1 - \sin^2 x - \sin^2 x \\ &= 1 - 2 \sin^2 x \end{aligned} \quad [23]$$

39. In formula [17] substitute x for y ,

$$\begin{aligned} \tan (x+x) &= \frac{\tan x + \tan x}{1 - \tan x \tan x} \\ \text{or} \quad \tan 2 x &= \frac{2 \tan x}{1 - \tan^2 x} \end{aligned} \quad [24]$$

*40. From formula [22], by transposing terms,

$$\begin{aligned} 2 \cos^2 x &= 1 + \cos 2 x \\ \text{or} \quad 2 \cos^2 \frac{1}{2} x &= 1 + \cos x \end{aligned} \quad [25]$$

*41. From formula [23], by transposing terms,

$$\begin{aligned} 2 \sin^2 x &= 1 - \cos 2 x \\ \text{or} \quad 2 \sin^2 \frac{1}{2} x &= 1 - \cos x \end{aligned} \quad [26]$$

42. Dividing [26] by [25],

$$\begin{aligned} \frac{2 \sin^2 \frac{1}{2} x}{2 \cos^2 \frac{1}{2} x} &= \frac{1 - \cos x}{1 + \cos x} \\ \tan^2 \frac{1}{2} x &= \frac{1 - \cos x}{1 + \cos x} \end{aligned}$$

* Formulas [25] and [26] are derived from the statement above each by substituting an angle equal to one-half the value of x

$$\tan \frac{1}{2} x = \sqrt{\frac{1 - \cos x}{1 + \cos x}} \quad [27]$$

43. From formulas [13] to [16] we obtain

$$\begin{aligned} \sin(x+y) + \sin(x-y) &= 2 \sin x \cos y \\ \sin(x+y) - \sin(x-y) &= 2 \cos x \sin y \\ \cos(x+y) + \cos(x-y) &= 2 \cos x \cos y \\ \cos(x+y) - \cos(x-y) &= -2 \sin x \sin y \end{aligned}$$

$$\text{Let } a = (x+y)$$

$$\text{and } b = (x-y)$$

Then by adding and subtracting we get

$$x = \frac{1}{2} (a+b) \text{ and } y = \frac{1}{2} (a-b)$$

Substituting,

$$\sin a + \sin b = 2 \sin \frac{1}{2} (a+b) \cos \frac{1}{2} (a-b) \quad [28]$$

$$\sin a - \sin b = 2 \cos \frac{1}{2} (a+b) \sin \frac{1}{2} (a-b) \quad [29]$$

$$\cos a + \cos b = 2 \cos \frac{1}{2} (a+b) \cos \frac{1}{2} (a-b) \quad [30]$$

$$\cos a - \cos b = -2 \sin \frac{1}{2} (a+b) \sin \frac{1}{2} (a-b) \quad [31]$$

These formulas greatly simplify the solution of oblique triangles in case two sides and the included angle are given, or if the three sides are known.

44. It can be proved that in any triangle

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad (\text{See Art 87.})$$

45. By Geometry (Composition and Division),

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}$$

46. By formulas [28] and [29],

$$\begin{aligned} \frac{a+b}{a-b} &= \frac{2 \sin \frac{1}{2} (A+B) \cos \frac{1}{2} (A-B)}{2 \cos \frac{1}{2} (A+B) \sin \frac{1}{2} (A-B)} \\ &= \frac{\sin \frac{1}{2} (A+B)}{\cos \frac{1}{2} (A+B)} \times \frac{1}{\frac{\sin \frac{1}{2} (A-B)}{\cos \frac{1}{2} (A-B)}} \\ &= \frac{\tan \frac{1}{2} (A+B)}{\tan \frac{1}{2} (A-B)} \quad [32] \end{aligned}$$

This formula is used when a , b and C of a triangle are given.

47. By advancing the letters,

$$\frac{b+c}{b-c} = \frac{\tan \frac{1}{2}(B+C)}{\tan \frac{1}{2}(B-C)}$$

$$\frac{c+a}{c-a} = \frac{\tan \frac{1}{2}(C+A)}{\tan \frac{1}{2}(C-A)}$$

48.

In Fig. 15

$$AD = b \cos (180 - A) \\ b \cos A$$

$$DB = AB + AD \\ c - b \cos A$$

$$CD = b \sin A$$

Use formulas [10] and [11].

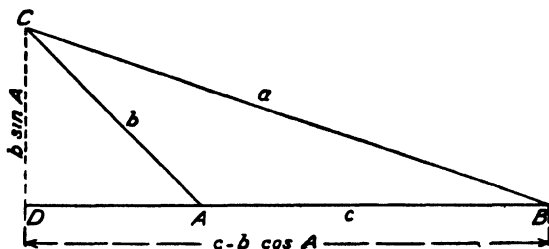


Fig. 15.

In Fig. 16

$$AD = b \cos A$$

$$DB = AB - AD \\ c - b \cos A$$

$$CD = b \sin A$$

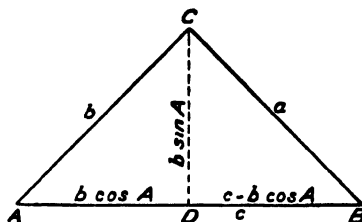


Fig. 16.

49. Now in both Fig. 15 and Fig 16

$$BC^2 = BD^2 + DC^2$$

$$\begin{aligned} a^2 &= (c - b \cos A)^2 + (b \sin A)^2 \\ &= c^2 - 2bc \cos A + b^2 \cos^2 A + b^2 \sin^2 A \\ &= c^2 - 2bc \cos A + b^2 (\cos^2 A + \sin^2 A) \\ &= c^2 + b^2 - 2bc \cos A \end{aligned}$$

$$\text{or } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

50. The above equation would enable us to find angle A if the three sides are given, but it is not in a form suitable for use with logarithms.

51. From formula [26]

$$\begin{aligned}
 2 \sin^2 \frac{1}{2} A &= 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} \\
 &= \frac{2bc - b^2 - c^2 + a^2}{2bc} \\
 &= \frac{a^2 - (b - c)^2}{2bc} \\
 &= \frac{(a - b + c)(a + b - c)}{2bc}
 \end{aligned}$$

Let $2s = a + b + c$, then $(a - b + c) = 2(s - b)$,
and $(a + b - c) = 2(s - c)$

$$\text{Substituting, } 2 \sin^2 \frac{1}{2} A = \frac{2(s - b)(s - c)}{bc}$$

$$\text{Hence } \sin \frac{1}{2} A = \sqrt{\frac{(s - b)(s - c)}{bc}} \quad [33]$$

52. From formula [25],

$$\begin{aligned}
 2 \cos^2 \frac{1}{2} A &= 1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc} \\
 &= \frac{2bc + b^2 + c^2 - a^2}{2bc} \\
 &= \frac{(b + c)^2 - a^2}{2bc} \\
 &= \frac{(b + c + a)(b + c - a)}{2bc} \\
 &= \frac{2s(s - a)}{bc}
 \end{aligned}$$

$$\text{Hence } \cos \frac{1}{2} A = \sqrt{\frac{s(s - a)}{bc}} \quad [34]$$

53. Dividing [33] by [34],

$$\frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} = \tan \frac{1}{2} A = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}} \quad [35]$$

54. Similarly

$$\begin{aligned}
 \tan \frac{1}{2} B &= \sqrt{\frac{(s - c)(s - a)}{s(s - b)}} \\
 \tan \frac{1}{2} C &= \sqrt{\frac{(s - a)(s - b)}{s(s - c)}}
 \end{aligned}$$

These formulas are used to find the angles when the three sides are given.

55. In Fig. 15 and Fig. 16 of Art. 48 we see that if K is the area of the triangle,

$$\begin{aligned} 2K &= AB \times CD \\ &= bc \sin A \end{aligned} \quad [36]$$

$$\sin 2X = 2 \sin X \cos X \quad [19] (a)$$

$$\text{or } \sin X = 2 \sin \frac{1}{2} X \cos \frac{1}{2} X \quad (b)$$

$$\text{or } \sin A = 2 \sin \frac{1}{2} A \cos \frac{1}{2} A \quad (c)$$

and transposing formula 36,

$$\sin A = \frac{2K}{bc}$$

Substitute this value of $\sin A$ in [19] (c) then $\frac{2K}{bc} = 2 \sin \frac{1}{2} A \cos \frac{1}{2} A$ or $2K = 2bc \sin \frac{1}{2} A \cos \frac{1}{2} A$. From [33] and [34] substitute values of $\frac{1}{2} A$ and get

$$2K = 2bc \sqrt{\frac{(s-b)(s-c)}{bc}} \times \sqrt{\frac{s(s-a)}{bc}}$$

or simplifying by multiplying together and cancelling bc with $\sqrt{bc} \times \sqrt{bc}$ and also 2 on each side of the equation.

$$\text{Then } K = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{Then } \log K = \frac{1}{2} (\log s + \log (s-a) + \log (s-b) + \log (s-c)) \quad [37]$$

$$\text{By Page 23, } S = \frac{a+b+c}{3}$$

LOGARITHMS.

56. The logarithm of a number, N , is the exponent of the power to which a fixed number, called the base, must be raised in order to produce N .

Thus, if $3^4 = 81$, $\log_3 81 = 4$; or, in words, the logarithm of 81 to base 3 is 4.

What is $\log_2 8$?

We know $2^3 = 8 \therefore \log_2 8 = 3$

What is $\log_9 3$?

We know $9^{\frac{1}{2}} = 3 \therefore \log_9 3 = \frac{1}{2}$

EXAMPLES FOR PRACTICE.

- | | |
|-------------------------------|---------|
| 1. Find the $\log_3 27$. | Ans. 3. |
| 2. Find the $\log_{10} 100$. | Ans. 2. |
| 3. Find the $\log_4 64$. | Ans. 3. |

57. All logarithms of numbers which have the same base are said to belong to the same *system*. The system of which 10 is the base is called the common system. This is the system most frequently used in computation.

58. In any system, the logarithm of the base itself is 1.

For	$10^1 = 10$	$\therefore \log_{10} 10 = 1$
	$3^1 = 3$	$\therefore \log_3 3 = 1$
	$a^1 = a$	$\therefore \log_a a = 1$

59. In any system the logarithm of 1 is 0.

For	$10^0 = 1$	$\therefore \log_{10} 1 = 0$
	$3^0 = 1$	$\therefore \log_3 1 = 0$
	$a^0 = 1$	$\therefore \log_a 1 = 0$

60. Logarithms are used largely to facilitate computation in problems involving multiplication, division, involution and evolution of numbers. Their great usefulness depends on the four following principles.

61. In any system, the logarithm of the product of two or more numbers is equal to the sum of the logarithms of the separate numbers.

Assume	$a^x = m$ (1)	{	Then	{	$\log_a m = x$
And	$a^y = n$ (2)		by § 56		$\log_a n = y$

Multiplying equations (1) and (2),

$$\begin{aligned} & (a^x) (a^y) = m n \\ \text{Or} \quad & a^{x+y} = m n \\ & \therefore \log_a m n = x + y \end{aligned}$$

Substituting for x and y their values,

$$\log_a m n = \log_a m + \log_a n$$

62. In any system the logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.

$$\begin{array}{ll} \text{Assume} & a^x = m \text{ (1) } \left\{ \begin{array}{l} \text{Then } \log_a m = x \\ \text{And } a^y = n \text{ (2) } \end{array} \right\} \text{ by § 56 } \left\{ \begin{array}{l} \log_a n = y \end{array} \right. \end{array}$$

Divide equation (1) by equation (2),

$$\frac{a^x}{a^y} = \frac{m}{n}$$

$$\text{Or} \quad a^{x-y} = \frac{m}{n}$$

$$\text{Then} \quad \log_a \frac{m}{n} = x - y$$

Substituting for x and y their values,

$$\log_a \frac{m}{n} = \log_a m - \log_a n$$

63. In any system the logarithm of the power of a number is equal to the exponent of the power multiplied by the logarithm of the number.

$$\text{Assume} \quad a^x = m \text{ (1), then } \log_a m = x.$$

Raising both members of (1) to the p^{th} power

$$a^{px} = m^p$$

Therefore by definition

$$\log_a (m^p) = p x$$

Substituting for x its value,

$$\log_a (m^p) = p \log_a m$$

64. The logarithm of a root of a number is equal to the logarithm of the number divided by the index of the root.

$$\text{By algebra,} \quad \sqrt[s]{m} = m^{\frac{1}{s}}$$

Therefore, by Art. 63,

$$\log_a \sqrt[s]{m} = \log_a m^{\frac{1}{s}} = \frac{1}{s} \log_a m = \frac{\log_a m}{s}$$

65. A few examples will illustrate these principles.

Given that $\log 2 = .301030$, $\log 3 = .477121$, $\log 5 = .698970$ and $\log 7 = .845098$.

(1) Find $\log 45$.

$$\begin{aligned} \log 45 &= \log (3 \times 3 \times 5) \\ &= \log 3 + \log 3 + \log 5 \\ &= 2 \times \log 3 + \log 5 \\ &= .954242 + .698970 \\ &= 1.653212. \end{aligned}$$

(2) Find $\log \frac{14}{5}$.

$$\begin{aligned} \log \frac{14}{5} &= \log 14 - \log 5 \\ &= (\log 2 + \log 7) - \log 5 \\ &= .301030 + .845098 - .698970 \\ &= 1.146128 - .698970 = .447158 \end{aligned}$$

(3) Find $\log (7^{\frac{3}{4}} \times 2^{\frac{1}{2}})$.

$$\begin{aligned} \log (7^{\frac{3}{4}} \times 2^{\frac{1}{2}}) &= \log 7^{\frac{3}{4}} + \log 2^{\frac{1}{2}} \\ &= \frac{3}{4} \log 7 + \frac{1}{2} \log 2 \\ &= \frac{3 \times (.845098)}{4} + \frac{.301030}{2} \\ &= .633824* + .150515 \\ &= .784339 \end{aligned}$$

(4) Find $\log \frac{\sqrt[4]{5}}{\sqrt[3]{2}}$.

$$\begin{aligned} \log \frac{\sqrt[4]{5}}{\sqrt[3]{2}} &= \log \sqrt[4]{5} - \log \sqrt[3]{2} \\ &= \frac{1}{4} \log 5 - \frac{1}{3} \log 2 \\ &= \frac{.698970}{4} - \frac{.301030}{3} \\ &= .174743 - .100343 \\ &= .074400 \end{aligned}$$

NOTE. Whenever in dividing a logarithm by a number a remainder of $\frac{1}{2}$ or more is obtained, the last figure of the quotient is increased by 1.

EXAMPLES FOR PRACTICE.

Find the logarithms of the following :

(5) $\log 35$	Ans. 1.544068.
(6) $\log 42$	Ans. 1.623249.
(7) $\log \frac{25}{7}$	Ans. .552842.
(8) $\log \sqrt{15}$	Ans. .588046.
(9) $\log (2^{\frac{2}{3}} \times 5^{\frac{1}{2}})$	Ans. .550172.
(10) $\log (7\sqrt[4]{6})$	Ans. 1.104482.
(11) $\log \sqrt[3]{50}$	Ans. .372364.

66. We will now write the common logarithms of certain integral powers of 10 and indicate between what limits the logarithms of certain other numbers must lie.

$1000 = 10^3$	$\therefore \log 1000 = 3$
$796 = 10^2 + \text{a decimal}$	$\therefore \log 796 = 2 + \text{a decimal}$
$100 = 10^2$	$\therefore \log 100 = 2$
$53 = 10^1 + \text{a decimal}$	$\therefore \log 53 = 1 + \text{a decimal}$
$10 = 10^1$	$\therefore \log 10 = 1$
$8 = 10^0 + \text{a decimal}$	$\therefore \log 8 = 0 + \text{a decimal}$
$1 = 10^0$	$\therefore \log 1 = 0$
$0.77 = 10^{-1} + \text{a decimal}$	$\therefore \log 0.77 = 9 + \text{a decimal} - 10$
$0.1 = \frac{1}{10} = 10^{-1}$	$\therefore \log 0.1 = -1 \text{ or } 9 - 10$
$0.079 = 10^{-2} + \text{a decimal}$	$\therefore \log 0.079 = 8 + \text{a decimal} - 10$
$0.01 = \frac{1}{100} = 10^{-2}$	$\therefore \log 0.01 = -2 \text{ or } 8 - 10$
$0.00274 = 10^{-3} + \text{a decimal}$	$\therefore \log 0.00274 = 7 + \text{a decimal} - 10$
$0.001 = \frac{1}{1000} = 10^{-3}$	$\therefore \log 0.001 = -3 \text{ or } 7 - 10$

67. From Art. 66 we see that the logarithm of a number which is not an integral power of 10 is an integer plus a decimal. Thus, 796 is between 10^2 and 10^3 , therefore its logarithm is between 2 and 3, and is $2 + \text{a decimal}$. Also, 0.079 is between 10^{-2} and 10^{-1} , therefore its logarithm is between -2 and -1 and is $-2 + \text{a decimal}$. But -2 is represented by $8 - 10$.

so that this logarithm would be written $8 + \text{a decimal} - 10$. Certain advantages are gained by using $8 - 10$ instead of $- 2$, as will be explained later on.

68. The integral part of the logarithm is called its *characteristic*, and the decimal part its *mantissa*. The characteristic, when the number is not an integral power of 10, may be found by observing between what adjacent integral powers of 10 the number lies.

69. From Art. 66 we see that the logarithm of a number between

1 and 10 is equal to $0 + \text{a decimal}$
 10 and 100 is equal to $1 + \text{a decimal}$
 100 and 1000 is equal to $2 + \text{a decimal}$, etc.

Therefore, the characteristic of the logarithm of a number with *one* figure to the left of the decimal point is 0; with *two* figures to the left of the decimal point is 1, with *three* figures to the left of the decimal point is 2, etc.

From this we may formulate the following rule:

The characteristic of the logarithm of a number greater than 1 is one less than the number of places to the left of the decimal point.

70. From Art. 66 we see that the logarithm of a number between

1 and .1 is equal to $9 + \text{a decimal} - 10$
 .1 and .01 is equal to $8 + \text{a decimal} - 10$
 .01 and .001 is equal to $7 + \text{a decimal} - 10$, etc.

Therefore, the characteristic of the logarithm of a decimal with no ciphers between the decimal point and the first significant figure is 9, with a $- 10$ written after the mantissa; of a decimal with one cipher between the decimal point and the first significant figure is 8, with a $- 10$ written after the mantissa; of a decimal with two ciphers between the decimal point and the first significant figure is 7, with a $- 10$ written after the mantissa.

From this we may formulate the following rule:

The characteristic of the logarithm of a number less than 1 is found by subtracting from 9 the number of ciphers between the decimal point and the first significant figure, writing a $- 10$ after the mantissa.

71. The mantissae of logarithms of numbers are given in

tables of logarithms, and we shall next take up the use of these tables. For the use and application of such tables the student is asked to study the Six-Place Logarithmic Table in the back of this book.

72. In the common system, the mantissas of the logarithms of numbers having the same sequence of figures are the same.

Suppose that $\log 2.405 = 0.381115$.

$$\begin{aligned}\text{Then} \quad \log 240.5 &= \log (2.405 \times 100) \\ &= \log 2.405 + \log 100 \\ &= 0.381115 + 2 \\ &= 2.381115\end{aligned}$$

$$\begin{aligned}\text{And} \quad \log .002405 &= \log (2.405 \times .001) \\ &= \log 2.405 + \log .001 \\ &= 0.381115 + 7 - 10 \\ &= 7.381115 - 10\end{aligned}$$

In general, if a number be multiplied or divided by an integral power of 10, the resulting number will have the same sequence of figures, and hence the logarithms of the two numbers will have the same mantissas.

73. In the use of logarithms it is necessary to perform the operations of addition, subtraction, multiplication and division. Certain peculiarities in the work will now be explained.

(1) **Addition.** If, after any addition, — 10, — 20, or — 30, etc., is written after the mantissa, and the characteristic is greater than 9, subtract from both parts of the characteristic such a multiple of 10 as will make the characteristic before the mantissa less than 10.

Thus:

$$\begin{aligned}9.368457 &- 10 \\ 8.236425 &- 10 \\ 17.604882 &- 20 \\ \text{or} \quad 7.604882 &- 10\end{aligned}$$

(2) **Subtraction.** In subtracting a greater logarithm from a less, or in subtracting a negative logarithm from a positive one, the characteristic of the minuend should be increased by 10 (or a multiple of 10), — 10 (or — the same multiple of 10) being written after the mantissa.

Thus, to subtract 3.429863 from 2.563298, we arrange the work as follows :

$$\begin{array}{r} 12.563298 - 10 \\ 3.429863 \\ \hline 9.133435 - 10 \end{array}$$

Also in subtracting 9.327891 — 10 from 4.679231, we arrange the work ;

$$\begin{array}{r} 14.679231 - 10 \\ 9.327891 - 10 \\ \hline 5.351340 \end{array}$$

(3) **Multiplication.** After a negative logarithm has been multiplied by an integer, subtract from both parts of the logarithm such a multiple of 10 as will leave — 10 written after the mantissa.

Thus, $2 \times (9.526438 - 10) = 19.052876 - 20$, which should be written $9.052876 - 10$.

(4) **Division.** Before dividing a negative logarithm by an integer we add to both parts of the logarithm such a multiple of 10 that after the division a — 10 will appear after the mantissa.

Thus,

$$\begin{array}{r} 8.429164 - 10 \\ \div 3 \\ \hline 28.429164 - 30 \\ \div 3 \\ \hline 9.476388 - 10 \end{array}$$

74. Solution of Arithmetical Problems.

(1) Find the value of $1200 \times .02561 \times 76.964$.

$$\log (1200 \times .02561 \times 76.964) = \log 1200 + \log .02561 + \log 76.964.$$

$$\begin{array}{r} \log 1,200 = 3.079181 \\ \log .02561 = 8.408410 - 10 \\ \log 76.964 = 1.886287 \\ \hline \log \text{Ans.} = 13.373878 - 10 \\ \text{or} \quad = 3.373878 \\ \therefore \text{Ans.} = 2365.26 \end{array}$$

Computation.

$$\begin{array}{r}
 \text{Mantissa } 7696 = 886265 \\
 \text{Tabular diff.} = 56 \quad 22 \\
 \quad .4 \quad \overline{886287} \\
 \quad \underline{22.4}
 \end{array}$$

$$\begin{array}{r}
 373878 \\
 \text{Next less mant. } 373831 - \text{No} = 2365 \\
 \text{Tab. diff.} = 184 \overline{47} \text{ (.255)} \\
 \quad 368 \\
 \quad \underline{1020} \\
 \quad 920 \\
 \quad \underline{1000}
 \end{array}$$

(2) Find the value of $\frac{336.842}{7980.2}$.

$$\log \frac{336.842}{7980.2} = \log 336.842 - \log 7980.2$$

$$\begin{array}{rcl}
 \log 336.842 = 12.527426 - 10 & \text{Mantissa } 3368 = 527372 \\
 \log 7980.2 = 3.902014 & \text{Tab. diff.} = 129 \quad 54 \\
 \log \text{Ans.} = 8.625412 - 10 & \quad .42 \quad \overline{527426} \\
 & \quad \underline{54.18}
 \end{array}$$

$$\text{Ans.} = .0422097$$

$$\text{Mantissa } 7980 = 902003$$

$$\begin{array}{r}
 11 \\
 \text{Tab. diff.} = 54 \quad 902014 \\
 \quad .2 \\
 \quad \underline{10.8}
 \end{array}$$

$$\begin{array}{r}
 625412 \\
 \text{Next less mant.} = 625312 - \text{No} = 4220 \\
 \text{Tab. diff.} = 103 \overline{100} \text{ (.97)}
 \end{array}$$

$$\begin{array}{r}
 927 \\
 \underline{730} \\
 721 \\
 \underline{\quad}
 \end{array}$$

(3) Find the value of $(77954)^4$.

$$\begin{array}{rcl}
 \log (7.7954)^4 & = & 4 \log 7.7954 \\
 \log 7.7954 & = & 0.891838 \\
 & & \underline{4} \\
 \log \text{Ans.} & = & 3.567352 \\
 \text{Ans.} & = & 3692.762
 \end{array}
 \qquad
 \begin{array}{rcl}
 \text{Mantissa } 7795 & = & 891816 \\
 \text{Tab. diff.} & = & 56 \qquad \underline{22} \\
 & & .4 \quad 891838 \\
 & & \underline{22.4}
 \end{array}$$

$$\begin{array}{rcl}
 & & 567352 \\
 \text{Next less mantissa} & 567262 - \text{No} & = 3692 \\
 \text{Tab. diff.} & = 118 \overline{)90.} & (.762 \\
 & & 819 \\
 & & \underline{810}
 \end{array}$$

(4) Find the value of $\sqrt[3]{.03511}$.

$$\begin{aligned}
 \log \sqrt[3]{.03511} &= \frac{1}{3} \log .03511 = \frac{8.545431 - 10}{3} \\
 &= \frac{28.545431 - 30}{3} \\
 \log \text{Ans.} &= 9.515144 - 10 \\
 \text{Ans.} &= .327449
 \end{aligned}$$

EXAMPLES FOR PRACTICE.

Calculate by logarithms the value of the following:

- (1) $(84.759) \times (2280.76)$. Ans. 193,315.
- (2) $(964.38)^{\frac{4}{3}}$. Ans. 244.004.
- (3) $4^{\frac{1}{2}} \times 7^{\frac{2}{3}}$. Ans. 13.8289.
- (4) $\frac{\sqrt{.08}}{(10)^{\frac{2}{3}}}$. Ans. .00710469.
- (5) $\frac{3.89612 \times .6946}{4694.9 \times .00454}$. Ans. .126965.

THE RIGHT TRIANGLE.

75. In the solution of triangles by logarithms we find it necessary to use the logarithms of the trigonometric functions of angles. These logarithms can be found in our six-place tables of Trigonometric Functions.

NOTE: The problems of the Trigonometry are solved by six-place tables,

so your answers will vary slightly from those given in the sixth figure of numbers and in seconds of angles.

A few examples will be worked out here.

76. (a) Find the $\log \sin 82^\circ 17' 26''$.

$$\begin{array}{rcl} \log \sin 82^\circ 17' & = & 9.996049 - 10 & D.1'' = .28 \\ & & \underline{7} & \underline{26} \\ \log \sin 82^\circ 17' 26'' & = & 9.996056 - 10 & \underline{168} \\ & & & \underline{56} \\ & & & 7.28 \end{array}$$

(b) Find the $\log \cot 13^\circ 52' 40''$.

$$\begin{array}{rcl} \log \cot 13^\circ 52' & = & 0.607553 & D.1'' = 9.03 \\ & & \underline{361} & \underline{40} \\ \log \cot 13^\circ 52' 40'' & = & 0.607192 & 361.20 \end{array}$$

(c) Find the angle whose $\log \tan = 9.385490 - 10$.

$$\begin{array}{rcl} & 9.385490 - 10 & \\ \text{Next less } \log \tan & = 9.385337 - 10 & [\text{angle} = 13^\circ 39'] \\ D.1'' & = 9.18)153. & (16.66 \\ & \underline{918} & \\ & 6120 & \\ & \underline{5508} & \\ & 612 & \end{array}$$

Hence the angle $= 13^\circ 39' 16.7''$.

(d) Find the angle whose $\log \cos = 9.742512 - 10$,

$$\begin{array}{rcl} \text{Next greater } \log \cos & = & 9.742652 - 10 \\ & 9.742512 - 10 & [\text{angle} = 56^\circ 26'] \\ D.1'' & = 3.17)140. & (44.16 \\ & \underline{1268} & \\ & 1320 & \\ & \underline{1268} & \\ & 520 & \\ & 317 & \\ & 2030 & \end{array}$$

Hence the angle $= 56^\circ 26' 44.2''$.

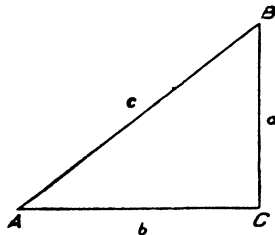
EXAMPLES FOR PRACTICE.

- (1) Find $\log \sin 30^\circ 17' 8''$. Ans. 9.702698 — 10.
 (2) Find $\log \cos 47^\circ 35' 16''$. Ans. 9.828956 — 10.

- (3) Find $\log \tan 75^\circ 40' 22''$. Ans. 0.592774.
 (4) Find $\log \cot 12^\circ 3' 45''$. Ans. 0.670202.
 (5) Given $\log \sin x = 9.983827 - 10$, find x .
Ans. $74^\circ 27' 37.9''$.
 (6) Given $\log \cos x = 9.877165 - 10$, find x .
Ans. $41^\circ 5' 35.5''$.
 (7) Given $\log \tan x = 0.248200$, find x .
Ans. $60^\circ 32' 51.2''$.
 (8) Given $\log \cot x = 1.169376$, find x .
Ans. $3^\circ 52' 24.01''$.

SOLUTION OF RIGHT TRIANGLES.

77. A right triangle can be completely solved if two sides, or one side and an acute angle, are given, by using the following formulas :



- (1) $c^2 = a^2 + b^2$ $A + B = 90^\circ$.
 (2) $\sin = \frac{\text{opposite side}}{\text{hypotenuse}}$ (4) $\tan = \frac{\text{opposite side}}{\text{adjacent side}}$
 (3) $\cos = \frac{\text{adjacent side}}{\text{hypotenuse}}$ (5) $\cot = \frac{\text{adjacent side}}{\text{opposite side}}$

Of these formulas select one in which two of the given parts enter, and solve for the third part.

There are five cases which can arise.

Case I. Given the hypotenuse and an acute angle.

Case II. Given a side and the opposite acute angle.

Case III. Given a side and the adjacent acute angle.

Case IV. Given the two sides about the right angle.

Case V. Given the hypotenuse and another side.

In the following computations we shall use certain abbreviations. T. D. = tabular difference; mant. = mantissa; *n. l. m.* = next less mantissa; No. = corresponding number of four places.

78. Case I. Given $\left\{ \begin{array}{l} A = 54^\circ 28' \\ c = 125.6 \end{array} \right\}$; find B , a and b .

$$B = 90^\circ - (54^\circ 28') = 35^\circ 32'.$$

Our formulas are, $\sin A = \frac{a}{c}$ (1) and $\cos A = \frac{b}{c}$ (2)

Whence $\log a = \log c + \log \sin A$

and $\log b = \log c + \log \cos A$

$\log c = 2.098990$	009496	
$\log \sin A = 9.910506 - 10$	$n. l. m. = 009451$	$\text{No.} = 1022$
$\log a = 2.009496$	$T. D. = 425$	$45.0(105$
$a = 102.211$		
$\log c = 2.098990$	863298	
$\log \cos A = 9.764308 - 10$	$n. l. m. = 863263$	$\text{No.} = 7299$
$\log b = 1.863298$	$T. D. = 60$	$35.0(58$
$b = 72.9958$		

79. Case II. Given $\left\{ \begin{array}{l} A = 67^\circ 9' 23'' \\ a = .235867 \end{array} \right\}$; find B , b and c .

$$B = 90^\circ - (67^\circ 9' 23'') = 22^\circ 50' 37''.$$

Our formulas are, $\tan A = \frac{a}{b}$, and $\sin A = \frac{a}{c}$

Whence $b = \frac{a}{\tan A}$ (1) and $c = \frac{a}{\sin A}$ (2).

From (1),

$\log b = \log a - \log \tan A$	mant. 2358	= 372544
$\log a = 9.372667 - 10$	$T. D. = 184$	123
$\log \tan A = 0.375452$	.67	372667
$\log b = 8.997215 - 10$	123.28	
$b = .0993607$	$\log \tan 67^\circ 9' = 0.375317$	
	$D.1'' = 5.88$	135
	23	0.375452
	135.24	
	997215	
$n. l. m. = 997212$	$\text{No.} = 9936$	
$T. D. = 43$	$3. (07$	

From (2),

$$\begin{array}{rcl} \log c = \log a - \log \sin A & \log \sin 37^\circ 9' = 9.964507 - 10 \\ \log a = 9.372667 - 10 & D. 1'' = .88 & 20 \\ \log \sin A = 9.964527 - 10 & 23 & \hline & & 9.964527 - 10 \\ \log c = 9.408140 - 10 & 20.24 & \\ c = .255941 & & \end{array}$$

$$\begin{array}{r} 408140 \\ n. l. m. = 408070 - \text{No.} = 2559 \\ \text{T. D.} = 170)70(.41 \end{array}$$

80. Case III. Given a side and the adjacent acute angle, we first find the other acute angle and then solve like Case II.

81. Case IV. Given $\left\{ \begin{array}{l} a = 226.61 \\ b = 543.94 \end{array} \right\}$; find A, B and c.

Our formulas are $\tan A = \frac{a}{b}$ (1) and $c = \frac{a}{\sin A}$ (2).

From (1)

$$\begin{array}{rcl} \log \tan A = \log a - \log b & \text{Mant. } 2266 = 355260 \\ \log a = 2.355279 & \text{T. D.} = 192 & 19 \\ \log b = 2.735551 & .1 & \hline & & 355279 \\ \log \tan A = 9.619728 - 10 & 19.2 & \\ A = 22^\circ 37' 1.3'' & \text{Mant. } 5439 = 735519 \\ \therefore B = 67^\circ 22' 58.6'' & \text{T. D.} = 80 & 32 \\ & .4 & \hline & & 735551 \\ & 32.0 & \end{array}$$

$$\begin{array}{r} 9.619728 \\ n. l. \log \tan = 9.619720 - \text{angle} = 22^\circ 37' \\ D. 1'' = 5.93) 8. (1.3 \end{array}$$

From (2),

$$\begin{array}{rcl} \log c = \log a - \log \sin A & & \\ \log a = 2.355279 & \log \sin 22^\circ 37' = 9.584968 - 10 \\ \log \sin A = 9.584975 - 10 & D. 1'' = 5.07 & 7 \\ \log c = 2.770304 & 1.4 & \hline & & 9.584975 - 10 \\ c = 589.256 & 7.098 & \end{array}$$

$$\begin{array}{r} 770304 \\ n. l. m. = 770263 - \text{No.} = 5892 \\ \text{T. D.} = 73)41(.56 \end{array}$$

82. Case V. Given $\left\{ \begin{array}{l} c = 47.65 \\ a = 21.34 \end{array} \right\}$; find A, B and b.

Our formulas are $\sin A = \frac{a}{c}$ (1), and $b = c \cos A$ (2).

From (1)

$$\log \sin A = \log a - \log c$$

$$\log a = 1.329194 \qquad 9.651131$$

$$\log c = 1.678063 \quad n. l. \log \sin = 9.651044 - \text{angle} = 26^\circ 36'.$$

$$\log \sin A = 9.651131 - 10 \quad D.1'' = 4.22 \overline{)87.} (20.6$$

$$A = 26^\circ 36' 20.6''.$$

From (2)

$$\log b = \log c + \log \cos A \qquad \log \cos 26^\circ 36' = 9.951412$$

$$\log c = 1.678063 \qquad D.1'' = 1.05 \qquad 22$$

$$\log \cos A = 9.951390 - 10 \qquad 20.6 \qquad 9.951390$$

$$\log b = 11.629453 - 10 \qquad 21.630$$

$$\therefore b = 42.6042 \qquad 629453$$

$$n. l. m. = 629410 - No. = 42.60$$

$$T. D. = 102 \overline{)43.} (.42$$

83. A table of natural sines and cosines is sometimes used in solving triangles. We will next solve the example of Case V by this method.

$$\sin A = \frac{a}{c} = \frac{21.34}{47.65}$$

$$= .44785$$

$$44785$$

$$n. l. \sin = 44776 - \text{angle} = 26^\circ 36' 9.$$

$$\therefore A = 26^\circ 36' 20.8'' \quad D.1'' = 26 \therefore D.1'' = \frac{26}{60}$$

$$\text{correction} = 9 \div \frac{26}{60} = 9 \times \frac{60}{26} = 20.8.$$

$$\text{Since } \cos A = \frac{b}{c}$$

$$\cos 26^\circ 36' = .89415$$

$$\therefore b = c \cos A$$

$$D.1' = 13 \qquad 5$$

$$\overline{.89410}$$

$$= 47.65 \times .89410$$

$$\text{cor.} = \frac{13}{60} \times 20.8 = 4.51$$

$$= 42.603865$$

$$\therefore b = 42.6039$$

EXAMPLES FOR PRACTICE.

Solve the following right triangles :

(1) Given $A = 15^\circ$, $c = 7$. Ans. $a = 1.8117$, $b = 6.7615$.

(2) Given $A = 38^\circ$, $a = 8.09$.

Ans. $b = 10.3547$, $c = 13.1404$.

(3) Given $A = 72^\circ 52'$, $b = 6.306$.

Ans. $a = 20.4556$, $c = 21.4056$

(4) Given $a = 529.3$, $c = 902.7$.

Ans. $A = 35^\circ 53' 55.2''$, $b = 731.237$.

(5) Given $a = 3.414$, $b = 2.875$.

Ans. $A = 49^\circ 53' 55''$, $c = 4.46330$.

(6) Given $A = 26^\circ 12' 24''$, $c = 469422.7$.

Ans. $a = 207302$, $b = 421,170$.

(7) Given $B = 82^\circ 6' 18''$, $a = 89.32$.

Ans. $b = 644.109$, $c = 650.272$.

Check the work on each of the above problems See Art. 85.

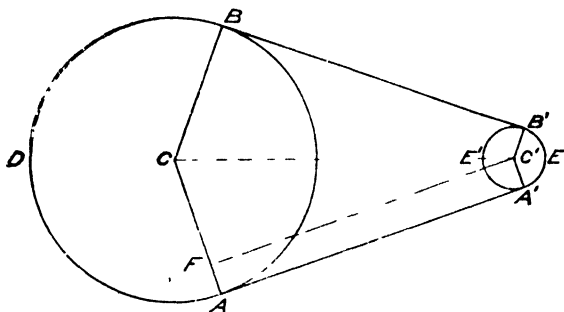


Fig. 17.

84. Problem. Find the length of a belt which passes around two pulleys, the radii of which are 6 feet and 1 foot respectively, and the distance between centers 13 feet.

In Fig. 17, let $CC' = 13$ feet, $CA = 6$ feet and $C'A' = 1$ foot. Draw $C'F$ parallel to AA' . Then $CF = 6 - 1 = 5$ feet. In the right triangle $CF C'$, $CC' = 13$, and $CF = 5$.

$$\begin{aligned} \text{Therefore, } C'F &= \sqrt{CC'^2 - CF^2} \\ &= \sqrt{169 - 25} = \sqrt{144} = 12. \end{aligned}$$

Let us now find the magnitude of angle FCC' in triangle FCC'

$$\sin FCC' = \frac{12}{13}$$

$$\begin{aligned}\log \sin FCC' &= \log 12 - \log 13 \\ &= 1.079181 - 1.113943 \\ &= 9.965238 - 10\end{aligned}$$

Therefore $\angle FCC' = 67^\circ 22' 48''.8$

$$\text{Twice } \angle FCC' = \angle ACB = 134^\circ 45' 37''.6$$

Subtracting this last result from 360° , we get the angle subtended by the arc ADB.

$$360^\circ - (134^\circ 45' 37''.6) = 225^\circ 14' 22''.4$$

By Geometry, arcs of the same circle are to each other as the subtended angles,

$$\begin{aligned}\text{Hence } \frac{\text{arc ADB}}{\text{circumference of the circle}} &= \frac{225^\circ 14' 22''.4}{360^\circ} \\ \text{or } \frac{\text{arc ADB}}{2\pi R} &= \frac{225.2396}{360} \quad \begin{array}{l} (11' 22.4'' = .2396^\circ) \\ (\pi = 3.1416) \end{array}\end{aligned}$$

$$\text{Then } 2\pi R = 2 \times 3.1416 \times 6 = 37.6992$$

$$\text{arc ADB} = \frac{37.6992 \times 225.2396}{360}$$

By logarithms

$$\log \text{arc ADB} = \log 37.6992 + \log 225.2396 - \log 360$$

$$\log 37.6992 = 1.576332$$

$$\begin{aligned}\log 225.2396 &= 1.352644 \\ &\quad 2.928976\end{aligned}$$

$$\log 36 = 1.556303$$

$$\log \text{arc ADB} = 1.372673$$

$$\text{arc ADB} = 23.587 \text{ ft.}$$

$$\text{Similarly } \frac{\text{arc A'E'B'}}{\text{circum. small circle}} = \frac{225.2396}{360}$$

$$\frac{\text{arc A'E'B'}}{2 \times 3.1416 \times 1} = \frac{225.2396}{360}$$

$$\log \text{arc } A'E'B' = \log 6.2832 + \log 22.52396 - \log 86$$

$$\log 6.2832 = 0.798181$$

$$\log 22.52396 = 1.352644$$

$$\hline 2.150825$$

$$\log 36 = 1.556303$$

$$\log \text{arc } A'E'B' = .594522$$

$$\therefore \text{arc } A'E'B' = 3.93117$$

But arc $A'EB'$ is the portion covered by the belt. Then

$$\text{arc } A'EB' = 6.2832 - 3.93117 = 2.35203$$

$\therefore$ the length of the belt equals

$$\text{arc } BDA + AA' + \text{arc } B'EA' + BB'$$

$$= 23.587 + 12 + 2.35203 + 12 = 49.939 \text{ feet.}$$

CHECK ON SOLUTIONS.

85. When a triangle has been solved, it is always desirable to know that the results are correct. This may always be determined by writing a formula that will involve all the parts found by solution, and some of the known parts if necessary. If both sides of the formula balance when the values of the quantities are substituted we conclude that our work is correct.

If, in the right triangle ABC , we are given $\angle A$ and side a , by solution we must find b and c . The formula for checking is

$$\cos A = \frac{b}{c}$$

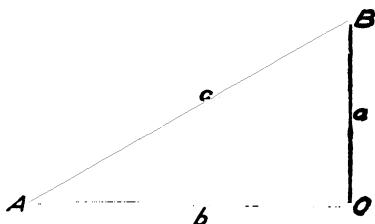
When given a and b , the formula for checking is

$$\sin A = \frac{a}{c}, \text{ or } \cos A = \frac{b}{c}$$

For example $a = 43.148, b = 84.107$. To find $\angle A$ and side c .

$$\tan A = \frac{43.148}{84.107}$$

$$\log \tan A = \log 43.148 - \log 84.107$$



TRIANGLES

$$\begin{aligned}
 \log 43.148 &= 1.634961 \\
 \log 84.107 &= 1.924832 \\
 \hline
 \log \tan A &= 9.71013 - 10 \\
 A &= 27^\circ 9' 30'' .5
 \end{aligned}$$

$$\cos A = \frac{b}{c}, \quad c = \frac{84.107}{\cos (27^\circ 9' 30'' .5)}$$

$$\begin{aligned}
 \log 84.107 &= 1.924832 \\
 \log \cos (27^\circ 9' 30'' .5) &= 9.949267 - 10 \\
 \hline
 \log c &= 1.975569 \\
 \therefore c &= 94.5289
 \end{aligned}$$

Check

$$\sin A = \frac{a}{c}$$

$$\sin (27^\circ 9' 30'' .5) = \frac{43.148}{94.53}$$

$$\begin{aligned}
 \log 43.148 &= 1.634961 \\
 \log 94.5289 &= 1.975565 - 10 \\
 \hline
 \text{Diff.} &= 9.659396 - 10
 \end{aligned}$$

$$\text{But } \log \sin (27^\circ 9' 30'' .5) = 9.659396$$

$$\therefore \log \sin (27^\circ 9' 30'' .5) = \log \frac{43.148}{94.53}$$

For the most accurate checking, each part should be found independent of the others. But in the above problem to have calculated c by the expression $c = \sqrt{a^2 + b^2}$ would have been laborious. Since c was found by means of $\cos A$, while the value of A was determined through $\tan A$, and in the check we used $\sin A$, we have practically used independent functions.

EXAMPLES FOR PRACTICE.

Solve the following right triangles, checking your answers in each case.

1. $a = 853$, $b = 904.75$
2. $b = .58752$, $A = 59^\circ 12'.3$
3. $c = .02584$, $b = .01967$
4. $c = 58.371$, $A = 69^\circ 53'.7$
5. $c = .84952$, $B = 55^\circ 59'.4$

HEIGHTS AND DISTANCES.

When an observer is stationed at A, as in Fig. 18, the $\angle CAB$ is called the angle of elevation of the point B. If the observer is stationed at B, the $\angle ABK$ is called the angle of depression of A. In practical work these angles are read by a sextant, theodolite or transit.

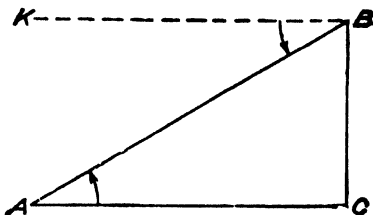


Fig. 18.

1. Suppose we wish to find the height of a hill, where A of Fig. 19 represents the highest point. From C, a point in the same level with the base of the hill, $\angle ACB$ is read 30° . A line CD in the same level is measured back 500 ft. and $\angle ADB$ read 20° . Let $AB = x$, and $CB = y$.

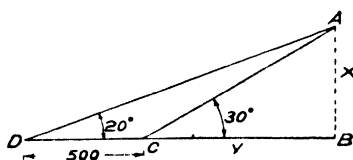


Fig. 19.

$$\tan 30^\circ = \frac{x}{y} \quad \tan 20^\circ = \frac{x}{y + 500}$$

$$\tan 30^\circ = .577, \quad y = \frac{x - 182}{.364}$$

By comparison

$$\frac{x}{.577} = \frac{x - 182}{.364} \quad .364x = .577x - 105$$

$$\therefore x = 493 \text{ ft.}$$

2. From the top of a cliff which is known to be 1,000 feet high, the angles of depression of two boats in the same straight line are $25^\circ 35'$ and $38^\circ 12'$. How far apart are the boats?

3. A tower 75 feet high has a pole surmounting it. From a point in the same plane with the base, but 200 feet away, the angles of elevation of the top and bottom of the pole are 32° and $20^\circ 33' 22''$ respectively. Find the length of the pole.

4. The foundations of two buildings are on the same level. From the bottom and top of the first building, the angles of elevation of the second building are 38° and 28° respectively. The first building is 72 feet high. How high is the second, and how far apart are the two buildings?

SOLUTION OF OBLIQUE TRIANGLES.

86. There are four cases of oblique triangles.

Case I. Given one side and two angles.

Case II. Given two sides and the angle opposite one of them.

Case III. Given two sides and the included angle.

Case IV. Given the three sides.

87. Cases I and II can be most easily solved by using the following formula:

The sides of a triangle are proportional to the sines of the opposite angles.

Let ABC be the given triangle.

In each figure the perpendicular, CD , comes in two right triangles.

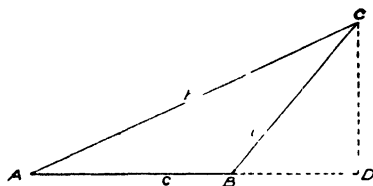


Fig. 19a.

$$\text{In one} \qquad \qquad \qquad CD = b \sin A$$

$$\text{In the other} \qquad \qquad \qquad CD = a \sin B$$

$$\text{Therefore} \qquad \qquad \qquad a \sin B = b \sin A$$

$$\text{Or} \qquad \qquad \qquad \frac{a}{b} = \frac{\sin A}{\sin B} \qquad [38]$$

$$\text{By drawing } \perp^s \text{ from the other vertices, the same relations may be shown} \qquad \frac{b}{c} = \frac{\sin B}{\sin C} \qquad [39]$$

$$\text{to hold between any pair of angles and sides.} \qquad \frac{a}{c} = \frac{\sin A}{\sin C} \qquad [40]$$

88. Case I. Given $A = 40^\circ 20'$, $B = 57^\circ 30'$, $b = 76$

$$\begin{aligned} \text{Then} \qquad \qquad \qquad C &= 180^\circ - (A + B) \\ &= 82^\circ 10' \end{aligned}$$

We have from Art. 87

$$\frac{a}{b} = \frac{\sin A}{\sin B} \quad (1) \quad \text{and} \quad \frac{c}{b} = \frac{\sin C}{\sin B} \quad (2)$$

From (1)

From (2)

$\log a = \log b + \log \sin A - \log \sin B,$ $\log b = 1.880814$ $\log \sin A = 9.811061$ $\quad = 1.691875$ $\log \sin B = 9.926029$ $\log a = 1.765846$ $a = 58.3238$	$\log c = \log b + \log \sin C - \log \sin B$ $\log b = 1.880814$ $\log \sin C = 9.995928$ $\quad = 1.876742$ $\log \sin B = 9.926029$ $\log c = 1.950713$ $c = 89.2714$
---	--

A check on this problem will be to determine the angle C from the formula $\frac{b}{c} = \frac{\sin B}{\sin C}$. Then compare the value of C with $180 - (A + B)$.

$$\log \sin C = \log c + \log \sin B - \log b.$$

$$\log c = 1.950713$$

$$\log \sin 57^\circ 30' = 9.926029$$

$$\underline{1.876742}$$

$$\log b = 1.880814$$

$$\log \sin C = 9.995928$$

$$\therefore C = 82^\circ 10' = 180 - (A + B) = C$$

Another and more reliable check is the formula

$$\frac{c + a}{c - a} = \frac{\tan \frac{1}{2}(C + A)}{\tan \frac{1}{2}(C - A)}$$

$$c = 89.2714 \qquad A = 40^\circ 20'$$

$$a = 58.3238 \qquad C = 82^\circ 10'$$

$$\underline{a + c = 147.5952} \qquad C + A = 122^\circ 30'$$

$$c - a = 30.9476 \qquad C - A = 41^\circ 50'$$

$$\frac{1}{2}(C + A) = 61^\circ 15'$$

$$\frac{1}{2}(C - A) = 20^\circ 55'$$

$$\log 147.5952 = 2.169071 \qquad \log \tan 61^\circ 15' = 0.280729$$

$$\log 30.9476 = 1.490627 \qquad \log \tan 20^\circ 55' = 9.582286$$

$$\underline{0.678444}$$

$$\underline{0.678443}$$

or $0.678444 = 0.678443$ (check)

80. Case II. Given two sides and an angle opposite one side.

Suppose we are given sides b , a , and angle A , to solve the triangle. In Fig 20 let $AC = b$, and draw AK making $\angle CAK = \angle A$. Draw CD perpendicular to AK . By definition $\sin A = \frac{CD}{b}$. Then $CD = b \sin A$. If $a = CL$, which is less than CD , the triangle would be impossible. That is, if $a < b \sin A$, there would be no solution.

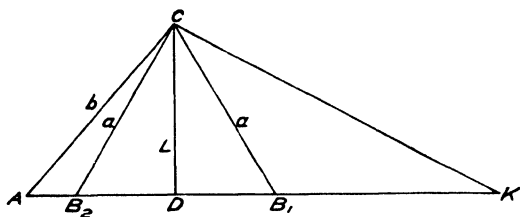


Fig. 20.

If $a < b \sin A = CD$, we have a right angled triangle ACD . If $a > b \sin A$, but less than b there are two triangles, CAB_2 and CAB_1 , with sides b , a and $\angle A$. If $a > b \sin A$ and $a > b$, we have the triangle ACK .

The results of this article are then

$$(a) \begin{cases} a < b \sin A, & \text{no solution.} \\ a = b \sin A, & \text{one solution (right angled triangle).} \\ b > a > b \sin A, & \text{two solutions.} \\ a > b > b \sin A, & \text{one solution.} \end{cases}$$

As an illustration of the principles of table (a) let us take up a few problems.

1. Given $A = 31^\circ$, $b = 19$, $a = 14$. To determine the number of solutions.

$$b \sin A = 19 \sin 31^\circ = 19 \times .515 = 9.785$$

Then $9.785 < 14 < 19$. That is $b \sin A < a < b$. There are then two solutions.

2. Given $B = 30^\circ$, $b = 10$, $c = 20$.

$$\text{Solution. } c \sin B = 20 \sin 30^\circ = 20 \times \frac{1}{2} = 10$$

Then $b = c \sin B$, and is equal to the perpendicular. We have but one solution, the right angled triangle.

3. Given $C = 37^\circ$, $c = 19$, $b = 7$.

Solution. $b \sin C = 7 \sin 37^\circ = 7 \times .602 = 4.214$

Since $c = 19 > 4.214$ and $19 > 7$, there will be one solution.

4. Given $A = 35^\circ$, $b = 25$, $a = 5$.

Solution. $b \sin A = 25 \sin 35^\circ = 25 \times .574 = 14.35$

Here $a = 5 < 14.35$ the perpendicular. The triangle is impossible. To make the conclusions more evident, construct each of the triangles given above, by means of protractor, compass and ruler.

90. A problem will here be solved to illustrate Case II.

Given $B = 21^\circ 10'$, $b = 42.73$, $c = 55.81$

By the law of sines

$$\sin C = \frac{c \sin B}{b}$$

$$\log \sin C = \log c + \log \sin B - \log b$$

$$\log c = 1.746712$$

$$\log \sin B = 9.557606 - 10$$

$$\hline 11.304318 - 10$$

$$\log b = 1.630733$$

$$\log \sin C = 9.673585 - 10$$

$$C = 28^\circ 8' 20''.3$$

$$\text{or } 180^\circ - (28^\circ 8' 20''.3) = 151^\circ 51' 39''.7$$

as we saw in Art. 27, two supplementary angles always have the same sine.

Now $b < c$, and from plane geometry B must be less than C . Either of the values of C satisfy this condition. We then have two triangles to satisfy the conditions of the problem. We may find side a when $C = 28^\circ 8' 20''.3$ by using the formula

$$\frac{a}{c} = \frac{\sin A}{\sin C}$$

$$A = 180 - (B + C) = 130^\circ 41' 39''.7$$

$$\log c = 1.746712$$

$$\log \sin A = 9.879783 - 10$$

$$\text{By add.} \quad \hline 1.626495$$

$$\log \sin C = 9.673585$$

$$\text{By sub. } \log a = \overline{1.952910}$$

$$a = 89.724$$

Check. $\frac{a + b}{a - b} = \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}$

$$\frac{132.454}{46.994} = \frac{\tan (75^\circ 55' 49''.9)}{\tan (54^\circ 45' 49''.9)}$$

$$\log 132.454 = 2.122065$$

$$\log 46.994 = 1.672042$$

$$\hline 0.450023$$

$$\log \tan (75^\circ 55' 49''.9) = 0.600992$$

$$\log \tan (54^\circ 45' 49''.9) = \overline{0.150969}$$

$$\hline 0.450023$$

$$\text{or } 0.450023 \quad 0.450023 \text{ (check)}$$

The value $C = 151^\circ 51' 39''.7$ will give a different value for a , which may be checked by the same formula as used in the preceding solution.

91. Case III. Two sides and the included angle; *First Method*:

$$\text{Given } b = 35.31 \text{ ft. } c = 42.18 \text{ ft. and } A = 31^\circ 26'$$

$$\text{By (32) } \frac{c + b}{c - b} = \frac{\tan \frac{1}{2}(C + B)}{\tan \frac{1}{2}(C - B)}$$

$$C + B = 180 - A = 180 - (31^\circ 26') = 148^\circ 34'$$

$$\frac{1}{2}(C + B) = 90 - \frac{1}{2}A$$

$$\tan \frac{1}{2}(C + B) = \tan (90 - \frac{1}{2}A) = \cot \frac{1}{2}A = \cot (15^\circ 43')$$

$$c + b = 77.49$$

$$c - b = 6.87$$

$$\frac{77.49}{6.87} = \frac{\cot (15^\circ 43')}{\tan \frac{1}{2}(C - B)}$$

$$\log \tan \frac{1}{2}(C - B) = \log 6.87 + \log \cot (15^\circ 43') - \log 77.49$$

$$\log 6.87 = 0.836957$$

$$\log \cot (15^\circ 43') = 0.550674$$

$$1.387631$$

$$\log 77.49 = 1.889246$$

$$\log \tan \frac{1}{2} (C - B) = 9.498385$$

$$\frac{1}{2} (C - B) = 17^\circ 29' 14''$$

$$\frac{1}{2} (C + B) = 74^\circ 17'$$

$$\therefore C = 91^\circ 46' 14'' \text{ by addition}$$

$$B = 56^\circ 47' 46'' \text{ by subtraction}$$

As a check on our work, let us substitute in formula

$$\frac{b}{c} = \frac{\sin B}{\sin C}$$

$$\text{Then } \log b - \log c = \log \sin B - \log \sin C$$

$$1.547898 - 1.625107 = 9.922584 - 9.99793$$

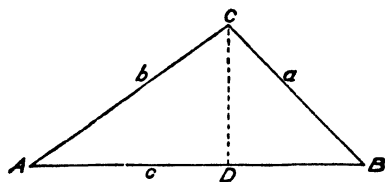
$$9.922791 - 10 = 9.922790 - 10$$

This result shows our work to be correct. The side a may now be found by the law of sines.

91. **Second Method.** (By right angled triangles.)

$$b = 35.31; c = 42.18; A = 31^\circ 26'.$$

From the right triangles we have the following:



$$(1) \quad CD = b \sin A$$

$$(2) \quad AD = b \cos A$$

$$(3) \quad DB = c - AD$$

$$(4) \quad \tan B = \frac{CD}{DB}$$

$$(5) \quad a = \frac{CD}{\sin B}$$

$$(6) \quad C = 180^\circ - (A + B)$$

From (1)

$$\begin{aligned} \log CD &= \log b + \log \sin A \\ \log b &= 1.547898 \\ \log \sin A &= 9.717259 - 10 \\ \log CD &= \overline{1.265157} \end{aligned}$$

From (2)

$$\begin{aligned} \log AD &= \log b + \log \cos A \\ \log \cos A &= 9.931075 - 10 \\ \log b &= 1.547898 \\ \log AD &= \overline{1.478973} \\ AD &= 30.1282 \\ DB &= 42.18 - 30.1282 \\ &= 12.0518 \end{aligned}$$

From (4)

$$\begin{aligned} \log \tan B &= \log CD - \log DB \\ \log CD &= 1.265157 \\ \log DB &= 1.081052 \\ \log \tan B &= \overline{0.184105} \\ B &= 56^\circ 47' 46''.1 \\ C &= 180^\circ - 88^\circ 13' 46''.1 = 91^\circ 46' 13''.9 \end{aligned}$$

From (5)

$$\begin{aligned} \log a &= \log CD - \log \sin B \\ \log CD &= 1.265157 \\ \log \sin B &= 9.922584 \\ \log a &= \overline{1.342573} \\ a &= 22.0076 \end{aligned}$$

Third Method. When given two sides and the included angle of a triangle it is frequently necessary to find the third side only. Either of the two methods given requires the finding of the remaining angles before finding the third side, which makes extra work. There is a formula given in art 49, namely

$$a^2 = b^2 + c^2 - 2bc \cos A$$

by means of which the third side may be found. The only objection to its use is that it does not admit of a direct application of logarithms. But if the sides b and c are not too large, and a alone is required, it is easily obtained by the given formula.

For example,

$$b = 35.31, c = 42.18, A = 31^\circ 26'$$

$$\begin{aligned} \log b &= 1.547898, & \log c &= 1.625107 \\ 2 \log b &= \log b^2 = 3.095796 & \log c^2 &= 3.250214 \\ b^2 &= 1246.8 & c^2 &= 1779.15 \end{aligned}$$

$$\begin{aligned} \log b &= 1.547898 \\ \log c &= 1.625107 \\ \log \cos A &= 9.931075 \\ \log 2 &= 0.301030 \\ \log 2 bc \cos A &= \overline{3.405110} \end{aligned}$$

$$2 bc \cos A = 2541.61$$

$$b^2 = 1246.8$$

$$c^2 = 1779.15$$

$$b^2 + c^2 = 3025.95$$

$$2 bc \cos A = 2541.61$$

$$b^2 + c^2 - 2bc \cos A = 484.34$$

$$a^2 = 484.34$$

$$\log a^2 = 2 \log a = 2.685150$$

$$\log a = 1.342575$$

$$a = 22.007$$

This problem might have been solved without logarithms, using actual multiplication and square root. Another objection that might be offered to the third method is, that we do not have a method for checking. Consequently, if one desires the strictest accuracy, the first method, although long, is the safest.

92. **Case IV.** (Three sides given.)

In articles 53 and 54 we find the following formulas :

$$\tan \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \quad \tan \frac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$$\tan \frac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

If we multiply numerator and denominator of $\tan \frac{1}{2} A$ by $\sqrt{s-a}$; $\tan \frac{1}{2} B$ by $\sqrt{s-b}$; $\tan \frac{1}{2} C$ by $\sqrt{s-c}$ we have

$$\tan \frac{1}{2} A = \frac{1}{s-a} \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \frac{R}{s-a}$$

$$\tan \frac{1}{2} B = \frac{1}{s-b} \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \frac{R}{s-b}$$

$$\tan \frac{1}{2} C = \frac{1}{s-c} \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \frac{R}{s-c}$$

In solving a problem, having given the three sides of the triangle, first determine $\log R$, tabulating all results carefully.

For example, given

$$a = 365, b = 422, c = 523$$

$$s = \frac{a + b + c}{2} = \frac{365 + 422 + 523}{2} = 655$$

$$s - a = 290$$

$$s - b = 233$$

$$s - c = 132$$

$$R = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$\log R = \frac{1}{2} [\log (s-a) + \log (s-b) + \log (s-c) - \log s]$$

$$\log 290 = 2.462398$$

$$\log \tan \frac{1}{2} B = \log R - \log (s-b)$$

$$\log 233 = 2.367356$$

$$= 9.699688 - 10$$

$$\log 132 = 2.120574$$

$$\frac{1}{2} B = 26^\circ 36' 10''.66$$

$$6.950328$$

$$B = 53^\circ 12' 21''.32$$

$$\log 655 = 2.816241$$

$$2 \overline{4.134087}$$

$$\log R = 2.067044$$

$$\log \tan \frac{1}{2} A = \log R - \log (s-c)$$

$$\log \tan \frac{1}{2} C = \log R - \log (s-a)$$

$$= 9.604646 - 10$$

$$= 9.946470 - 10$$

$$\frac{1}{2} A = 21^\circ 55' 9''.54$$

$$\frac{1}{2} C = 41^\circ 28' 40''.24$$

$$A = 43^\circ 50' 19''.08$$

$$C = 82^\circ 57' 20''.48$$

The check on this work is $A + B + C = 180$

°		"
43	50	19.08
53	12	21.32
82	57	20.48
180.	0	0.88

Which shows that the work is correct.

AREAS OF TRIANGLES.

93. The area of a triangle is one-half the product of its base by its altitude. Hence, if we denote the area of the triangle by K , the formula for the area of the right triangle in Art. 77 is

$$2K = ab.$$

This formula used in connection with those in Art. 77 will enable us to find the area of any right triangle.

94. In oblique triangles we may have three cases.

Case I. Given two sides and the included angle.

In Fig. 19a, Art. 87,

$$2 K = c \times CD.$$

But $CD = b \sin A.$

Hence $2 K = bc \sin A.$

Similarly $2 K = ab \sin C.$

$$2 K = ca \sin B$$

Case II. Given one side and the three angles.

From Case I,

$$2K = bc \sin A. \quad (1)$$

And from Art. 87,

$$\frac{b}{c} = \frac{\sin B}{\sin C}$$

$$\therefore b = \frac{c \sin B}{\sin C}.$$

Substituting in (1) for b the value just found.

$$2 K = \frac{c^2 \sin A \sin B}{\sin C}.$$

Similarly $2 K = \frac{a^2 \sin B \sin C}{\sin A}.$

And $2 K = \frac{b^2 \sin C \sin A}{\sin B}.$

Case III. Given the three sides.

Unless we use the formula [37] deduced in Art. 55, we must break up the oblique triangle into right triangles according to the method of Art. 92. It is better to use formula [37].

EXAMPLES FOR PRACTICE.

Solve the triangles, find their areas, and check all results;

- (1) Given $a = 180$, $A = 38^\circ$, $B = 75^\circ 43'$.
- (2) Given $c = 24.637$, $A = 83^\circ 39'$, $B = 38^\circ 56'$.
- (3) Given $a = 5.98$, $b = 3.59$, $A = 63^\circ 50'$.
- (4) Given $b = .41782$, $c = .38279$, $C = 29^\circ 46'$.
- (5) Given $a = 886$, $b = 747$, $C = 71^\circ 54'$.
- (6) Given $b = 4.102$, $c = 4.549$, $A = 62^\circ 9' 38''$.
- (7) Given $a = 5$, $b = 7$, $c = 6$.
- (8) Given $a = 10$, $b = 9$, $c = 8$.

ANSWERS.

- (1) $b = 283.331$, $c = 267.677$.
- (2) $a = 29.0595$, $b = 18.3742$.
- (3) $B = 32^\circ 36' 9''.4$, $c = 6.62085$.
- (4) $B_1 = 32^\circ 48' 47''.4$, $a_1 = .684403$.
- and $B_2 = 147^\circ 11' 12''.6$, $a_2 = .0409772$.
- (5) $A = 60^\circ 44' 39''.5$, $c = 965.282$.
- (6) $C = 63^\circ 49' 9''.3$, $a = 4.48237$.
- (7) $A = 44^\circ 24' 54''.8$, $B = 78^\circ 27' 47''.0$, $C = 57^\circ 7' 17''.6$.
- (8) $A = 71^\circ 47' 24''.4$, $B = 58^\circ 45' 5''.4$, $C = 49^\circ 27' 30''.0$.

TRIGONOMETRY

PART II.

INSTRUMENTS.

1. For the work of this course the student is asked to provide himself with a protractor, notebook, ruler, and dividers, or compasses.

The protractor is an instrument used for measuring angles, as well as for constructing angles of given magnitudes. It usually consists of a semicircle of cardboard, or metal, with the degrees marked on the circumference. To read the number of degrees in an angle, place the center of the protractor at the vertex, so that the diameter coincides with one of the boundary lines, or sides, of the angle. Then the number of degrees in the arc of the protractor, between the two boundary lines of the angle, represents the number of degrees in the angle. To construct a given angle on a given line, place the diameter of the protractor on the given line, with the center on the point where the vertex of the angle is to be located; also mark the point on the circumference where the required number of degrees is found. Join this point to the center of the protractor and we have the desired angle.

The *ruler* should be graduated to sixteenths and thirty-seconds of an inch for reasonably accurate work.

The *notebook* is to be used for keeping the results of the experiments performed. The object of such experiments is to have the student verify by hand and eye what theory has established, in this way gaining a better knowledge of the principles involved.

CIRCULAR MEASURE.

2. From the definition given in Article 17, Part I, any angle may be considered as a central angle of a circle whose radius is a segment of the initial line. The method for expressing angles in circular measure is then known to depend upon the two following propositions from plane geometry:

(a) In the same circle, or in equal circles, central angles are to each other as their intercepted arcs.

(b). The ratio of the circumferences of circles to their diameters is constant and is represented by $\pi = 3.1416$.

3. Construct an arc of a circle with the arbitrary radius OK as shown in Fig. 1. Choose arc BK equal in length to the radius OK , and draw OA and OB . From (a).

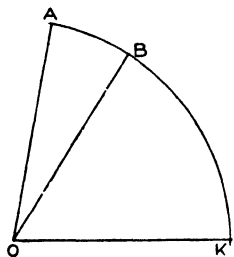


Fig 1.

$$\frac{\angle AOK}{\angle BOK} = \frac{\text{arc } AK}{\text{arc } BK} = \frac{\text{arc } AK}{r} \quad (1)$$

We will take $\angle BOK$ as a unit of measurement, and call it a *radian*. A *radian* is then a central angle whose intercepted arc is equal to the radius.

From (b). 180° of arc $= \pi r$ units of arc.

$$r \text{ units of arc} = \frac{(180)^\circ}{\pi} = 57.2958^\circ. \quad (2)$$

That is, a central angle whose intercepted arc is equal to the radius contains 57.2958° . But such an angle has been defined as

$$\text{a radian} \quad \text{Therefore } 1 \text{ radian} = \frac{(180)^\circ}{\pi} = 57.2958^\circ. \quad (3)$$

$$\text{or } \angle BOK = 57.2958^\circ.$$

Then from (3) $180^\circ = \pi$ radians.

$$1^\circ = \left(\frac{\pi}{180}\right) \text{ radians}. \quad (4)$$

When the number of degrees in an angle is divided by 57.2958° or when the arc that a central angle intercepts is divided by the radius the angle is said to be expressed in *radians* or *circular measure*.

Note. Hereafter 57.3 will be used as the number of degrees in a radian.

EXAMPLES FOR PRACTICE.

- Express 5° in circular measure.

$$\text{Solution: } 5^\circ = \frac{5}{57.3} = .087 \text{ radians,}$$

- Express $18^\circ 16' 25''$ in circular measure.

$$\text{Solution: } 16' 25'' = 985''$$

$$985'' = \frac{985'}{3600} = .274'$$

$$\text{Then } 18^\circ 16' 25'' = 18.274$$

$$\frac{18.274}{57.3} = .319 \text{ radians.}$$

3. Find the number of degrees in a central angle which intercepts a 5-inch arc from the circumference of a 7-inch circle.

Solution: From the definition of the circular measure of an angle $\theta = \frac{5}{7}$ radians. 1 radian = 57.3.

$$\angle \theta = \frac{5}{7} \times 57.3 = \frac{286.5}{7} = 40.93'$$

4. Express in circular measure correct to the third decimal place: 1° , $3^\circ 48'$, $12^\circ 56' 24''$, 42° , $65^\circ 11' 2''$.

5. Reduce to degrees, minutes and seconds the following angles which are expressed in radians:

$$.00254, .2693, \frac{1}{4}, \frac{1}{2}, \pi, 2 + \frac{1}{8}.$$

6. A train running at the rate of 40 miles per hour, is on a curve of $\frac{1}{2}$ mile radius. Through what angle will any part of the train pass in 15 seconds?

7. A train is running on a curve of $\frac{1}{2}$ mile radius. In 10 seconds the engine passes through $\frac{1}{8}$ of a radian. Find the rate of the train.

8. Two men run in the same direction on a circular track of radius 210 feet, one at the rate of 16 feet per second, and the other at 18 feet. At the end of $\frac{1}{2}$ hour how many degrees and minutes will the second be in advance of the first?

ANGLES IN TERMS OF π

4. Referring to equation (4), it is seen that when an angle is given in degrees, it may be expressed in terms of π radians by multiplying the number of degrees by $\frac{\pi}{180}$.

For example:

$$30^\circ = 30 \times \frac{\pi}{180} \text{ radians} = \frac{\pi}{6} \text{ rad}$$

$$90^\circ = 90 \times \frac{\pi}{180} \text{ radians} = \frac{\pi}{2} \text{ rad.}$$

Conversely when an angle is given in terms of π , it may be

reduced to degrees by multiplying the number of radians by $\frac{180}{\pi}$.

For example:

$$\frac{3\pi}{2} = \frac{3\pi}{2} \times \frac{180}{\pi} = 270^\circ.$$

It is customary to write π radians as π , the radians being understood.

1. Express the following angles in terms of π : 0° , 20° , 45° , 135° , 225° , 330° , 360° .

2. Reduce to degrees: $\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{5\pi}{6}$, $\frac{7\pi}{8}$, $\frac{9\pi}{5}$, $\frac{13\pi}{18}$.

ANGLES PLOTTED.

5. When the lengths of the arcs that are intercepted by central angles are computed and measured on a straight line from some fixed point, the angles are said to be plotted.

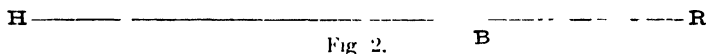
In equation (1) let $\angle AOK = \theta$. Then

(1) may be written:

$$\frac{\theta}{57.3} \text{ are A K}$$

$$\text{or} \quad \text{are A K} = \frac{r \cdot \theta}{57.3} = \frac{r \cdot \theta \cdot \pi}{180} \quad (5)$$

We use the number of degrees in the angle since in either form of (5) we divide degrees by degrees and obtain an abstract value. If $r = 5$ inches, then the arc of 50° equals $\frac{5 \cdot 50}{57.3} = 4.36$ inches.



Draw the straight line HR as in Fig. 2; choose H as the point from which we measure. Lay off HB equal to 4.36 inches. 50° is then said to be plotted on HR to radius or unit 5 inches.

When we are to plot several angles to the same unit, the problem is best solved by logarithms. In (5) let

$$\frac{r}{57.3} = R$$

Then

$$\text{arc } \theta = R \cdot \theta$$

$$\log \text{arc } \theta = \log R + \log \theta.$$

We would first compute $\log R$, and then $\log \text{arc } \theta$ is obtained by adding $\log \theta$. For example, when $r = 7$ inches

$$R = \frac{7}{57.3}$$

$$\log R = \log 7 - \log 57.3$$

$$\log 7 = .84510$$

$$\log 57.3 = 1.75816$$

$$\log R = .08694$$

$$\log \text{arc } 20^\circ = .08694 + \log 20$$

$$= .08694 + 1.30103 = .38797$$

$$\therefore \text{arc } 20^\circ = 2.443 -$$

$$\log \text{arc } 50^\circ = .08694 + 1.69894 = .78591$$

$$\text{arc } 50^\circ = 6.109 \text{ inches}$$

Draw a straight line. From one extremity measure off 2.443 and 6.109 inches and we have 20° and 50° plotted to unit 7 inches.

Example: Given $r = 4$ inches. Plot the angles 9° , 25° , 40° , 150° , 175° , 350° , 360° , 13° $19'.6$, 67° $18'.1$.

2. When $r = 6$ inches plot the same angles as in example 1.

6. When we are given the segment of a line with the unit of measurement, that is, the radius, the number of degrees that it equals may be found from equation (1), first reducing to circular measure and then to degrees. Also when given the length of a line with the number of degrees that it represents, we may easily find the radius of the circle by solving (5) for r . For example, if 8 inches represents 80 degrees, substituting in (5)

$$8 = \frac{r \times 80}{57.3}; r = \frac{8 \times 57.3}{80} = 5.73 \text{ inches.}$$

If 8 inches represented 100° ,

$$8 = \frac{r \times 100}{57.3}; r = \frac{8 \times 57.3}{100} = 4.584 \text{ inches.}$$

EXAMPLES FOR PRACTICE.

1. A line 6 inches long represents central angles of 125° ,

75° , 54° and 135° respectively. Find the radius of the circle, or unit of measurement in each case.

2. If the radius of a circle is 3 inches what is the largest angle in degrees that may be plotted on a 5-inch line?

CO-ORDINATE PAPER.

7. Angles are usually plotted on co-ordinate paper. Ordinary co-ordinate paper has two sets of parallel lines drawn at right angles to each other, on one side of the sheet. These lines are placed very close and at regular intervals. Those numbered 5 and 10 are drawn heavier than the others in order that they may be more easily recognized and counted.

When we use co-ordinate paper we adopt an arbitrary unit such as 2 degrees for one space, or 5 degrees for one space, depending entirely upon the length of the line on which we wish to plot the angles.

If we use 180 spaces to represent 360° , then 1 space equals 2 degrees. To plot 35° count $17\frac{1}{2}$ spaces along any arbitrarily chosen line (preferably the heavy one through the center of the page) beginning with the left extremity of the line; 100 spaces would stand for 200° , etc.

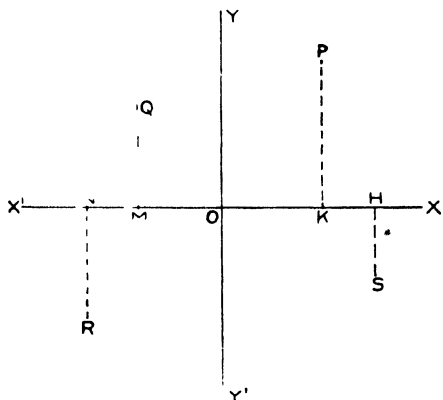


Fig. 3.

Exercise: Take a sheet of co-ordinate paper and plot angles every 10° from 0 to 360° , using 1 space for 2 degrees. Plot angles every 15° from 0 to 360° , using 1 space for 3 degrees.

CO-ORDINATES OF A POINT.

8. For the purpose of definitely locating every point in a plane we draw two lines XX' and YY' at right angles to each other, as shown in Fig. 3. These lines are known as axes of reference; O , the point of intersection, is known as the origin.

The axes divide the plane into four parts called quadrants.

The quadrants are numbered I, II, III and IV, beginning with the upper right hand one, and counting counter clockwise.

The distance of any point from the $Y Y'$ axis when taken with its direction is known as the abscissa of the point; the distance

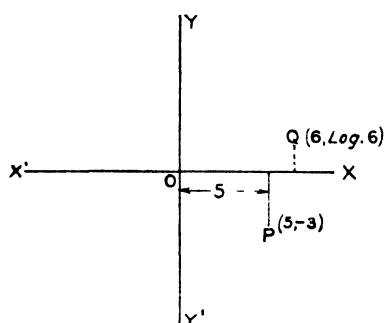


Fig. 1.

from the $X X'$ axis taken with the direction is called the ordinate. The abscissa and ordinate taken together are called the co-ordinates of the point. Abscissas measured to the right are positive and those to the left are negative. Ordinates measured up are positive, while those drawn down are considered negative.

The abscissa is usually represented by x , and the ordinate by y . The co-ordinates are written symbolically (x, y) . When the co-ordinates of a point are given, to plot the point we measure the abscissa on the x axis, and at its extremity we measure the ordinate at right angles. For example, to locate the point whose co-ordinates are $(5, -3)$, measure 5 units from 0 to the right on the x axis as in Fig. 4, then down 3 units, locating the point P.

Given the co-ordinates $(6 \log_{10} 6)$, to find the point.

We measure 6 units to the right from 0, and up to $\log_{10} 6 = .77815$, locating the point Q.

Given the co-ordinates $(20, \sin 20^\circ)$, to plot the point when the radius $r = 5$ inches.

$$\text{arc } 20^\circ = \frac{20 \times \pi}{180} \times 5 = \frac{3.1416 \times 5}{9} = \frac{15.708}{9} = 1.745 \text{ inches.}$$

$$\sin 20^\circ = .342.$$

Expressed in terms of r , $\sin 20^\circ = .342 \times 5 = 1.71$ inches.

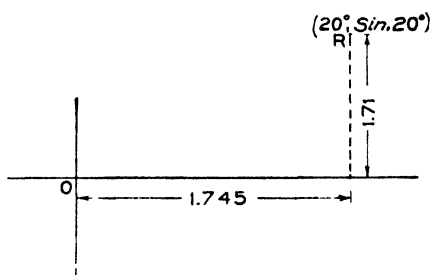


Fig. 5

Measuring the abscissa of 1.745 inches and ordinate 1.71 inches we find the point R.

EXAMPLES FOR PRACTICE.

1. Plot the points whose co-ordinates are:
 $(-2, -3)$, $(-5, 2)$, $(0, -7)$, $(8, 0)$.
2. Plot the points whose co-ordinates are:
 $(1, \log_{10} 1)$, $(5, \log_{10} 5)$, $(10, \log_{10} 10)$.
3. Given radius = 4 inches. Plot the points whose co-ordinates are:
 $(10, \sin 10^\circ)$, $(15, \cos 15^\circ)$, $(40, \tan 40^\circ)$, $(\frac{1}{2}, \sin \frac{1}{2})$, $(\frac{3}{4}, \tan \frac{3}{4})$
 $(1\frac{1}{2}, \cos 1\frac{1}{2})$, $(1, \sin (2 + \frac{1}{2}))$, $(\frac{\pi}{2}, \cos \frac{\pi}{2})$, $(\frac{5}{12}\pi, \tan \frac{5}{12}\pi)$.

FUNCTIONS FOR ALL ANGLES.

9. In Part I, article 4, it was shown that the trigonometric functions are independent of the length of the sides of the triangle, depending entirely upon the ratios of these sides. Again in article 20, Part I, it was shown that if from any point of the terminal line of an angle we drop a perpendicular to the initial line, we will form a right angled triangle, from which the functions of the angle may be written, no matter in what quadrant the angle may be found.

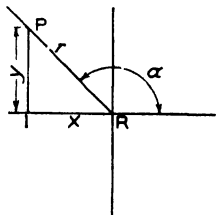


Fig 6.

In Fig. 6 an angle is represented in the second quadrant. The point P is chosen arbitrarily on the terminal line, from which the perpendicular P I is dropped. Then it is easily seen that R I and I P are the co-ordinates of P which are represented by y and x . The distance P R is represented by r . Then by definition

$$\sin a = \frac{y}{r}, \tan a = \frac{y}{x} \sec a = \frac{r}{x},$$

$$\cos a = \frac{x}{r}, \cot a = \frac{x}{y}, \operatorname{cosec} a = \frac{r}{y}.$$

Since r is distance, it is taken as positive. Then $\sin a$ will vary in sign with y . $\cos a$ will vary in sign with x . $\tan a$ and $\cot a$ will depend upon both x and y . They will be positive when x

and y are the same in sign, and negative when x and y are opposite.

The table of signs for the four quadrants as shown on Page 13, Part I, follows at once.

EXAMPLES FOR PRACTICE.

1. Write the signs of the functions for each of the following angles: 70° , 175° , 312° , $(\frac{3}{2}\pi + 50^\circ)$, $7(\frac{1}{4}\pi - 10^\circ)$, $\frac{5}{8}(2\pi + 60^\circ)$.
2. $\cot \theta = -\frac{3}{4}$, $\sin \theta$ is positive. In what quadrant is θ ?
3. Between what limits for $\theta < 360^\circ$ is $\sin \theta + \cos \theta$ positive, and for what limits is it negative?
4. Between what limits for $x < 360^\circ$ is $\sin x - \cos x$ positive? For what values negative? Using the table of values of article 25, Part I, find the values of the expressions:

$$5. \frac{5 \sin 90^\circ - 3 \cos 360^\circ}{\sec 180^\circ}$$

$$6. \frac{2 \operatorname{cosec} 70^\circ (\cot 90^\circ - 4 \sin 90^\circ)}{\cos 180^\circ - \sin 270^\circ}$$

GRAPHICAL REPRESENTATION OF THE TRIGONOMETRIC FUNCTIONS.

10. We saw in article 9 that for an angle in any quadrant, a point could be chosen in the terminal line at a constant distance from the center. The co-ordinates of this point together with the constant distance r form a triangle from which the functions of the angle could be written.

Let us choose r as a radius and describe the circle as in Fig. 7. Choose IC and FO as the axes of reference, with O the center of the circle, as the origin. Construct tangents to the circle at C and F . Draw IP and GO , producing them until they intersect the tangents at D , E , G and N . Let $\angle BOP = \theta_1$, $\angle BOH = \theta_2$, $\angle BOL = \theta_3$, and $\angle BOM = \theta_4$. Where the terminal lines of θ_1 , θ_2 , θ_3 , and θ_4 cut the circumference, drop the perpendiculars PB , HK , IL and MA .

Choose the radius of the circle as a unit of measurement. That is $r = OC = 1$.

Now by definition

$$\left. \begin{aligned} \sin \theta_1 &= \frac{BP}{OP} = \frac{BP}{r} = BP \\ \cos \theta_1 &= \frac{OB}{OP} = \frac{OB}{r} = OB \\ \tan \theta_1 &= \frac{BP}{OB} = \frac{CD}{OC} = \frac{CD}{r} = CD \\ \cot \theta_1 &= \cot \angle OEF = \frac{FE}{OF} = FE \end{aligned} \right\} \quad (6)$$

The circle of Fig. 7 is known as the unit circle. We are not to understand that the trigonometric functions are lines. The functions are always ratios. But if we choose our unit of meas-

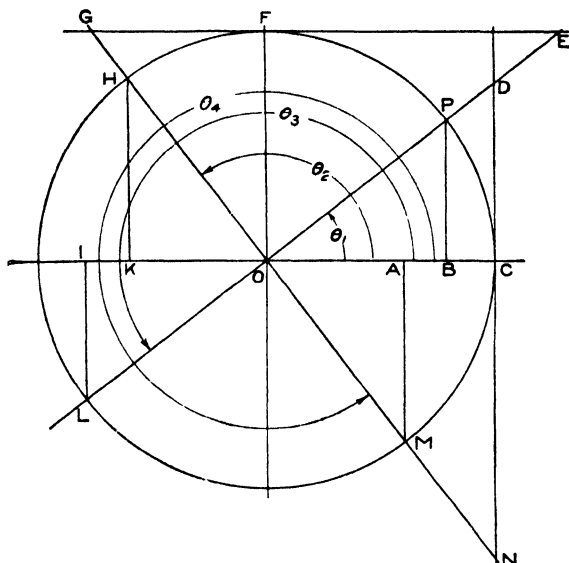


Fig. 7.

urement as the radius, the lines BP , OB etc. expressed in terms of this unit, may be taken to represent the functions.

From the unit circle of Fig. 7 we may write :

$\sin \theta$ is the ordinate of the point P .

$\cos \theta$ is the abscissa of the point P .

$\tan \theta$ is the segment of the tangent at C , included between the point of tangency and the terminal line.

$\cot \theta$ is the portion of the tangent at F included between the point of tangency and terminal line.

The preceding definitions are given when θ is in the first quadrant. For an angle θ_2 in the second quadrant of Fig. 7,

$$\sin \theta_2 = \frac{KH}{OH} = KH, \cos \theta_2 = \frac{OK}{OH} = OK, \tan \theta_2 = \frac{HK}{OK} = \frac{CN}{OC} = CN.$$

Here CN is drawn down and is therefore negative, as it ought to be, since the tangent in the second quadrant is negative.

$$\cot \theta_2 = \frac{OK}{HK} = \cot OGF = -\frac{FG}{OF} = FG.$$

FG is drawn to the left and is negative. We see by this that the definitions given in (6) are true for angles in the second quadrant.

Exercise. Determine the lines to represent the functions of θ_3 and θ_4 of Fig. 7 and show that the definitions of (6) hold true for angles in the third and fourth quadrants.

Exercise. Using the circle of Fig 7 trace each of the functions from 0° to 360° , and compare with the table of art. 25, Part I.

A FUNCTION OF AN ACUTE ANGLE.

II. The discussion on pages 16 and 17 of Part I, on the relation between functions of angles, may be summed up in the following law:

Express the angle as a multiple of 90° plus or minus an acute angle X . If the multiple is odd, take the complementary function of X ; if even the same function, and in either case prefix the sign of the original function as determined from the quadrant in which the angle is located.

For example, find $\sin 245^\circ$. Solution: $\sin 245^\circ = \sin (180^\circ + 65^\circ) = -\sin 65^\circ$. 180° is an even multiple of 90° , therefore we take the $\sin 65^\circ$. We prefix the minus sign because 245° is in the third quadrant, and the sine of an angle in this quadrant is negative.

Or, $\sin (245^\circ) = \sin (270^\circ - 25^\circ) = -\cos 25^\circ$ because 270° is an odd multiple of 90° . Both results are correct because $\sin 65^\circ = \sin (90^\circ - 25^\circ) = \cos 25^\circ$.

1. In the following exercises, express each function as the function of an acute angle.

(a) $\sin 325^\circ$

(c) $\cos 128^\circ$

(e) $\sin 175^\circ$

(b) $\cot 165^\circ$

(d) $\tan 167^\circ$

(f) $\cos 200^\circ$

2. From a table of natural functions find

$\sin 358^\circ$

$\cot 290^\circ$

$\sin .654$

$\cot 2.45$

$\cos 243^\circ$

$\sin 175^\circ$

$\cos .032$

$\sin \frac{5}{3}$

$\tan 157^\circ$

$\cos 149^\circ$

$\tan 1.78$

$\cos \frac{5}{12}$

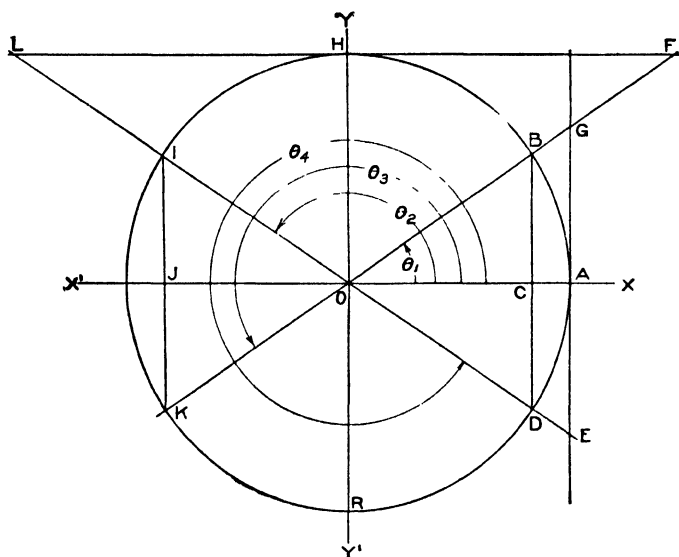


Fig. 8.

3. Simplify the following expressions:

(a) $\sin (90 + x) \cos (180 - y) + \cos (90 + x) \sin (180 - y)$

(b) $\tan \theta \frac{1}{\cot \theta} + \tan (180^\circ + \theta).$

4. Verify the following equalities:

(c) $\cos (90 + x) \cos (180 - x) + \sin (90 + x) \sin (180 + x) = 0$

(d) $\frac{\tan (180 - x)}{\tan (270 - x)} = \frac{\cot (90 + x)}{\cot (180 + x)}$

5. Find the values of the following expressions to the third decimal place:

$$\begin{aligned} 1.543 &\times \sin 168^\circ \\ .0063 &\times \tan 245^\circ \\ 145.26 &\cos (1.25 \text{ radians}) \end{aligned}$$

12. We know from plane geometry that we can draw but two equal chords perpendicular to a diameter, and these chords will be bisected by the diameter. In Fig. 8 let us choose the X axis as the diameter to which we will draw I K and B D two equal and perpendicular chords.

Then $C B = C D = j I, = j K$ in absolute value.

When these lines are taken as the ordinates of points, $C B = j I$ and $j K = C D$, the first pair being positive and the second pair negative. If we choose $O A$ as unity

$$\sin \theta_1 = C B, \sin \theta_2 = j I, \sin \theta_3 = j K, \sin \theta_4 = C D$$

Then $\sin \theta_1 = \sin \theta_2, \sin \theta_3 = \sin \theta_4$. Now from Fig. 8 $\angle B O H = \angle H O I$, and $\angle K O R = \angle R O D$

That is θ_1 and θ_2 differ by the same amount from 90° and θ_3 and θ_4 from 270° .

Therefore two angles with the same sine differ by the same amount from 90° and from 270° . (7)

From the graphical definition of cosine

$$O C = \cos \theta_1 = \cos \theta_4.$$

$$O j = \cos \theta_2 = \cos \theta_3$$

But $\angle B O A = \angle A O D$ and $\angle I O j = \angle j O K$

Then two angles with the same cosines differ by the same amount from 360° and 180° . (8)

Again

$$A G = \tan \theta_1 = \tan \theta_3$$

$$A E = \tan \theta_2 = \tan \theta_4$$

$$H F = \cot \theta_1 = \cot \theta_3$$

$$H L = \cot \theta_2 = \cot \theta_4$$

But θ_1 and θ_3 also θ_2 and θ_4 differ by 180° .

Therefore two angles with the same tangent or cotangent differ by 180° . (9)

EXAMPLES FOR PRACTICE.

1. Given $\sin \theta = \sin 52^\circ$. Find θ .

Solution: From (7), article 12, there will be two angles

which differ by the same amount from 90° . One value of θ is 52° since their sines are equal.

$$90 - 52 = 38, 90 + 38 = 128.$$

$$\therefore \theta = 52^\circ \text{ and } 128^\circ.$$

2. $\cos \theta = \cos 160^\circ$. Find both values of θ .

Now $\theta = 160^\circ$ since cosines are equal.

Applying (8)

$$180^\circ - 160^\circ = 20^\circ, 180^\circ + 20^\circ = 200^\circ$$

$$\angle \theta = 160^\circ \text{ and } 200^\circ$$

3. $\tan \theta = \tan 330^\circ$. Find θ .

One value of θ is 330° . From (9) the other value is found from

$$330^\circ - 180^\circ = 150^\circ.$$

$$\therefore \theta = 330^\circ \text{ and } 150^\circ.$$

4. $\sin \theta = -\sin 190^\circ$. Find θ

We will first apply the principles of article 10, in order to make the right hand member positive, with the same functions of the angles on both sides.

$$\sin \theta = -\sin (180^\circ + 10^\circ) = \sin 10^\circ.$$

We take $\sin 10^\circ$, since 180° is an even multiple of 90° . The

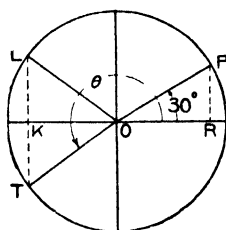


Fig. 9.

sign changes, since 190° is in the third quadrant, and sine is negative.

Then $\theta = 10^\circ$, and 170° , since each differ by the same amount from 90° .

5. $\cos^2 \theta = \sin^2 240^\circ$. Find all values of θ .

$$\cos \theta = \pm \sin 240^\circ.$$

Considering the positive sign first,

$$\cos \theta = \sin 240^\circ = \sin (270^\circ - 30^\circ) = -\cos 30^\circ$$

Construct a unit circle as shown in Fig. 9. Draw $\angle P O R = 30^\circ$. Then $O R = \cos 30^\circ$. $O K = -\cos 30^\circ$.

But $\angle K = \cos \angle LOR$ and $\cos \angle TOR$

$$\angle LOR = 180^\circ - 30^\circ = 150^\circ$$

$$\angle TOR = 180^\circ + 30^\circ = 210^\circ$$

Then $\theta = 150^\circ$ and 210°

If $\cos \theta = -\sin 240^\circ = -\sin (270^\circ - 30^\circ) = \cos 30^\circ$

$$\theta = 30^\circ \text{ and } 360^\circ - 30^\circ = 330^\circ \text{ from (8)}$$

$\therefore \theta = 30^\circ, 150^\circ, 210^\circ, \text{ and } 330^\circ.$

Note. A graphical solution of each of these problems will be helpful.

Find θ in each of the following problems:

6. $\sin \theta = \cos 175^\circ.$

7. $\cos \theta = \cos 68^\circ.$

8. $\tan \theta = \tan 160^\circ.$

9. $\sin \theta = \sin 290^\circ.$

10. $\cos \theta = -\cos 250^\circ.$

11. $\tan \theta = \tan 320^\circ.$

12. $\cot \theta = \cot 165^\circ.$

13. $\cot \theta = -\cot 300^\circ.$

14. $\sin 2\theta = -\sin 14^\circ.$

15. $\cot \frac{1}{2}\theta = -\tan 145^\circ.$

16. $\tan^2 \theta = \cot^2 65^\circ.$

TO EXPRESS ONE FUNCTION IN TERMS OF EACH OF THE OTHERS.

13. In article 12 we found that there were always two angles with the same trigonometric function, each angle being in a different quadrant. Then when the function of an angle is given, the angle is not completely determined unless the quadrant is also mentioned. By means of the first eight formula of Part I, we may then write all the remaining functions of our angle.

1. Given $\cot x = -\frac{3}{4}$, x in the fourth quadrant. Find the other functions of x .

$$\text{Solution. } \tan x = \frac{1}{\cot x} = \frac{1}{-\frac{3}{4}} = -\frac{4}{3}$$

$$\csc^2 x = 1 + \cot^2 x = 1 + \frac{9}{16} = \frac{25}{16}$$

Since $\csc x$ is negative in the fourth quadrant we choose the negative value of the square root. Then

$$\csc x = -\frac{5}{4}$$

$$\text{Now } \sin x = \frac{1}{\csc x} = \frac{1}{-\frac{5}{4}} = -\frac{4}{5}$$

$$\sec^2 x = 1 + \tan^2 x = 1 + \left(\frac{4}{3}\right)^2 = \frac{25}{9}$$

$\sec x = \frac{5}{3}$, since $\sec x$ is positive in the fourth quadrant

$$\cos x = \frac{1}{\sec x} = \frac{1}{\frac{5}{3}} = \frac{3}{5}$$

In each of the following problems find the remaining functions, as shown in problem;

2. $\cos x = -\frac{2}{3}$, x in quadrant III.

3. $\sin x = \frac{2}{5}$, x in quadrant II.

4. Versin $x = \frac{3}{5}$, x in quadrant III. (See Art. 23)

5. $\tan x = -\frac{5}{12}$, x in quadrant IV.

6. $\cot x = \frac{7}{2}$, x in quadrant III.

7. $\sec x = -\frac{1}{2}$, x in quadrant III

8. If $\cos \theta = \sqrt{\frac{s(s-a)}{bc}}$, where $s = \frac{a+b+c}{2}$,

show that $\sin \theta = \sqrt{\frac{(s-b)(s-c)}{bc}}$

Each of the preceding problems admit of the following geometrical solution: Given $\cot \theta = -\frac{3}{4}$, θ in quadrant IV.

Construct an angle θ in the fourth quadrant as shown in Fig. 10.

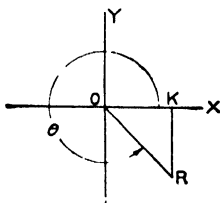


Fig. 10.

By definition $\cot \theta = \frac{OK}{KR}$. Then OK may be taken as 3, KR as 4, and $OR = 5$. By definition

$$\sin \theta = \frac{KR}{OR} = \frac{4}{5}, \cos \theta = \frac{OK}{OR} = \frac{3}{5}$$

$$\tan \theta = \frac{KR}{OK} = \frac{4}{3}, \sec \theta = \frac{5}{3}, \csc \theta = \frac{5}{4}$$

9. Solve 2, 3 and 4 of the preceding problems geometrically.

10. Given $\cos x = \frac{1}{3} \sec x$, when $x < 90$. Find all the functions of x .

Solution. Since $\sec x = \frac{1}{\cos x}$,

$$\cos x = \frac{1}{3} \frac{1}{\cos x}$$

$$\cos^2 x = \frac{1}{3}, \cos x = \pm \frac{1}{\sqrt{3}}$$

$$\sec x = \frac{1}{\cos x} = \pm \frac{1}{\frac{1}{\sqrt{3}}} = \pm \sqrt{3}$$

$$\sin x = \pm \sqrt{1 - \cos^2 x} = \pm \sqrt{1 - \frac{1}{3}} = \pm \frac{\sqrt{2}}{\sqrt{3}} = \pm \frac{\sqrt{6}}{3}$$

$$\csc x = \frac{1}{\sin x} = \pm \frac{1}{\frac{\sqrt{6}}{3}} = \pm \frac{3}{\sqrt{6}}$$

$$\tan x = \pm \frac{\sec x}{\cos x} = \pm \frac{\sqrt{3}}{\frac{1}{\sqrt{3}}} = \pm \sqrt{3}$$

$$\cot x = \frac{1}{\tan x} = \pm \frac{1}{\sqrt{3}}$$

In the following equation, find all the functions of the angle.

11. $2 \sin \theta = 3 \cos \theta$

12. $\tan \theta = 5 \cos \theta$

13. $\cot \Lambda = \frac{1}{2} \csc \Lambda$

14. $\cos \theta = \frac{5}{4} \tan \theta$.

15. Express $\sin x$ in terms of each of the other functions.

Solution: Since $\sin^2 x + \cos^2 x = 1$, by solving for $\sin x$ there results

$$\sin x = \pm \sqrt{1 - \cos^2 x}$$

By formula, $\cos x = \frac{1}{\sec x}$. Replacing $\cos x$ by $\frac{1}{\sec x}$ we have

$$\sin x = \pm \sqrt{1 - \frac{1}{\sec^2 x}} = \pm \sqrt{\frac{\sec^2 x - 1}{\sec^2 x}}$$

But $\sec^2 x = 1 + \tan^2 x$. Then

$$\sec x = \pm \sqrt{1 + \tan^2 x} \text{ and } \tan x = \pm \sqrt{\sec^2 x - 1}$$

$$\text{This gives } \sin x = \pm \frac{\tan x}{\sqrt{1 + \tan^2 x}}$$

Substituting $\frac{1}{\cot x}$ for $\tan x$

$$\sin x = \frac{\frac{1}{\cot x}}{\sqrt{1 + \frac{1}{\cot^2 x}}} = \frac{1}{\sqrt{1 + \cot^2 x}}$$

Also by definition $\sin x = \frac{1}{\csc x}$

Summing up the results of the above solution

$$\begin{aligned} \sin x &= \pm \sqrt{1 - \cos^2 x} = \pm \frac{\tan x}{\sqrt{1 + \tan^2 x}} = \pm \frac{1}{\sqrt{1 + \cot^2 x}} \\ &= \pm \frac{\sqrt{\sec^2 x - 1}}{\sec x} = \pm \frac{1}{\csc x} \end{aligned}$$

These results were not obtained in the order written, as the order is immaterial. In the following table the student is asked to show that each function in the first column is equal to every expression written on the same horizontal line, determining them in any order he pleases.

16.*

$\cos x$	$\frac{1 - \sin^2 x}{\cos x}$	$\frac{1}{1 + \tan^2 x}$	$\frac{\cot x}{1 + \cot^2 x}$	$\frac{1}{\sec x}$	$\frac{\sqrt{\csc^2 x - 1}}{\csc x}$
$\tan x$	$\frac{\sin x}{\sqrt{1 - \sin^2 x}}$	$\frac{1 - \cos^2 x}{\cos x}$	$\frac{1}{\cot x}$	$\frac{\sqrt{\sec^2 x - 1}}{\sec x}$	$\frac{1}{\csc^2 x - 1}$
$\cot x$	$\frac{1 - \sin^2 x}{\sin x}$	$\frac{\cos x}{1 - \cos^2 x}$	$\frac{1}{\tan x}$	$\frac{1}{\sec^2 x - 1}$	$\frac{\sqrt{\csc^2 x - 1}}{\csc x}$

*The double sign is understood to be written before each expression in the above table

17. Given $\tan 200^\circ = a$. Show that $\sin 290^\circ = -\frac{1}{\sqrt{1 + a^2}}$

Solution. $\tan 200^\circ = \tan (180^\circ + 20^\circ) = \tan 20^\circ = a$

Since the functions of an angle depend on the ratios of the sides, we may draw a right angled triangle with one acute angle

20° , in which the opposite side BC is a , and AC is unity. Then $AB = \sqrt{1 + a^2}$. Now $\sin 290^\circ = \sin (270^\circ + 20^\circ) = -\cos 20^\circ$. From the triangle

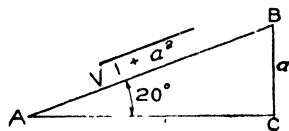


Fig. 10_a.

$$\cos 20^\circ = \frac{1}{\sqrt{1 + a^2}}$$

$$\sin 290^\circ = -\cos 20^\circ = -\frac{1}{\sqrt{1 + a^2}}$$

The method to be used in solving problems similar to example 17 is to express each function in terms of a function of the same acute angle. By means of the known function a triangle may be constructed, and the values of the required functions written by definition from this triangle.

18. If $\cos 140^\circ = -a$ prove that

$$\csc 320^\circ = \frac{\sqrt{1 - a^2}}{1 - a^2}$$

19. If $\tan 130^\circ = -c$ prove that

$$\frac{\sin 220^\circ \cot 320^\circ}{\sec 140^\circ} = \frac{c^2}{1 + c^2}$$

14. One of the most useful applications made of trigonometry, is the changing of an expression involving trigonometric functions from one form into another. In the following equations, one member is to be changed into the other by the use of the eight fundamental theorems on pages 9 and 10 of Part I.

$$1. \quad \tan x + \cot x = \sec x \cdot \csc x.$$

$$\text{Solution: } \tan x = \frac{\sin x}{\cos x}, \cot x = \frac{\cos x}{\sin x}$$

Substituting

$$\begin{aligned} \tan x + \cot x &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\cos x \cdot \sin x} \\ &= \frac{1}{\cos x \cdot \sin x} = \frac{1}{\cos x} \cdot \frac{1}{\sin x} = \sec x \cdot \csc x \end{aligned}$$

Since $\sin^2 x + \cos^2 x = 1$, $\sec x = \frac{1}{\cos x}$

and $\csc x = \frac{1}{\sin x}$

Then $\sec x \cdot \csc x = \sec x \cdot \csc x$.

$$2. \sec^2 x + \csc^2 x = \sec^2 x \cdot \csc^2 x.$$

Solution: Substituting $\frac{1}{\cos x}$ and $\frac{1}{\sin x}$ for $\sec x$ and $\csc x$ respectively we obtain the form

$$\begin{aligned} \sec^2 x + \csc^2 x &= \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cdot \cos^2 x} \\ &= \frac{1}{\sin^2 x \cdot \cos^2 x} = \frac{1}{\sin^2 x} \cdot \frac{1}{\cos^2 x} = \csc^2 x \cdot \sec^2 x \end{aligned}$$

Q. E. D.

$$3. \sin^4 x - \cos^4 x = 1 - 2 \cos^2 x.$$

Solution: Factoring the left hand member

$$\begin{aligned} \sin^4 x - \cos^4 x &= (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x) \\ &= (\sin^2 x - \cos^2 x), \text{ since } \sin^2 x + \cos^2 x = 1. \end{aligned}$$

But $\sin^2 x = 1 - \cos^2 x$. Then

$$\sin^2 x - \cos^2 x = 1 - \cos^2 x - \cos^2 x = 1 - 2 \cos^2 x$$

Q. E. D.

$$4. 1 - \sin x = \frac{\cos^2 x}{1 + \sin x}$$

(Hint. Work with right hand member)

$$5. \sec \theta - \tan \theta = \frac{1}{\sec \theta + \tan \theta}$$

$$6. \frac{\cot Z \cdot \cos Z}{\cot Z + \cos Z} = \frac{\cot Z - \cos Z}{\cot Z \cdot \cos Z}$$

$$7. \tan t + \cot t = \frac{\sec^2 t + \csc^2 t}{\sec t \cdot \csc t}$$

$$8. (\tan x + \cot x)^2 = \sec^2 x + \csc^2 x.$$

$$9. (1 + \tan A)(1 + \cot A) = \sec A \csc A + 2$$

$$10. \frac{\cos y \cdot \sin y}{\cos y + \sin y} = \frac{\cos y - \sin y}{\cot y - \tan y}$$

$$11. \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$$

$$12. (\tan x + \cot x)(\tan x - \cot x) = \sec^2 x - \csc^2 x.$$

13. $\left(\frac{\tan x}{\operatorname{cosec} x} + \cos x \right) = \sec x$.
14. $(\tan \theta + \cot \theta) \sin \theta \cos \theta = 1$
15. $(\sin^3 \theta + \cos^3 \theta) = (\sin \theta + \cos \theta) (1 - \sin \theta \cos \theta)$
(Factor left hand member of 15).
16. $\operatorname{cosec}^3 x (\tan x - \sin x) = \frac{1}{\cos x (1 + \cos x)}$
17. $\frac{\sec^2 Z}{\sin^2 Z} - 2 = \tan^2 Z + \frac{1}{\tan^2 Z}$
18. $\tan y + \cot y = \cos y \cdot \sin y (\sec^2 y + \operatorname{cosec}^2 y)$

SOLUTION OF EQUATIONS.

15. An equation involving trigonometric functions is called a trigonometric equation. In order to determine the values of the angle which satisfy the equation, we transform each member so that it will contain but one function of the angle. We then solve for this function, and from our knowledge of the values of functions for particular angles, or from tables of functions, we determine the values of $\theta < 360^\circ$.

Solve the following equations:

1. $4 \cos \theta = 3 \sec \theta$.

Substituting $\frac{1}{\cos \theta}$ for $\sec \theta$,

$$4 \cos \theta = 3 \frac{1}{\cos \theta}$$

or

$$4 \cos^2 \theta = 3$$

$$\cos^2 \theta = \frac{3}{4}$$

$$\cos \theta = \pm \frac{1}{2} \sqrt{3}$$

Now $\cos 150^\circ = \cos 210^\circ = -\frac{1}{2} \sqrt{3}$

$$\cos 30^\circ = \cos 330^\circ = +\frac{1}{2} \sqrt{3}$$

$\therefore \cos \theta = \cos 150^\circ = \cos 210^\circ$

and $\cos \theta = \cos 30^\circ = \cos 330^\circ$

$\therefore \theta = 30^\circ, 150^\circ, 210^\circ, 330^\circ$

2. $3 \sin \theta - 2 \cos^2 \theta = 0$
3. $\tan \theta + \cot \theta = 2$
4. $3 \tan^2 \theta = 1 + 4 \sin^2 \theta$
16. Prove that $\sin \theta < \theta < \tan \theta$. (θ in circular measure)

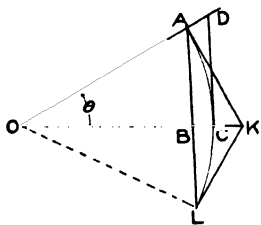


Fig. 11.

Construct the circular arc $A C L$, as in Fig. 11. Bisect $\angle A O L$ and let $\theta = \angle A O C$. Draw chord $A L$; $D C \perp$ to $O K$, and draw the tangents $A K$ and $K L$. Let the radius of the circle be adopted as a unit of measure.

From Geometry we know that chord $A B L < \text{arc } A C L < A K + K L$.
 $A B L < \text{arc } A C L < A K + K L$.
 Then $A B < A C < A K$. But $A K = C D$
 $\therefore A B < A C < C D$.

Since r is our unit of measure, $A B = \sin \theta$; $A C = \theta$ (circular measure), $D C = \tan \theta$.

$$\therefore \sin \theta < \theta < \tan \theta \quad (10)$$

As an illustration of this principle, reduce the angles to circular measure correct to fifth place, and write in the third column of the following table. Find the sines and tangents of these angles and tabulate the results as shown. Compare the results in order to determine the truth of the preceding principle.

TABLE A.

θ°	$\sin \theta$	θ in circular measure	$\tan \theta$
5			
2			
1			
$30'$			
$15''$			

Dividing (10) through by $\sin \theta$, we have $1 < \frac{\theta}{\sin \theta} < \sec \theta$

Now as θ approaches zero as a limit, symbolically written $\theta \rightarrow 0$,

$$\sec \theta \rightarrow 1, \text{ since } \cos \theta \rightarrow 1 \text{ and } \sec \theta = \frac{1}{\cos \theta}$$

Then $\frac{\theta}{\sin \theta}$ is greater than 1 and less than a quantity which can be made to differ from one by an amount less than any quantity we may choose to name.

$$\frac{\theta}{\sin \theta} \text{ or } \frac{\sin \theta}{\theta} \rightarrow 1 \quad (11)$$

Dividing (10) by $\tan \theta$ we have

$$\cos \theta < \frac{\theta}{\tan \theta} < 1$$

But $\cos \theta \rightarrow 1$ as $\theta \rightarrow 0$

$$\therefore \frac{\theta}{\tan \theta} \rightarrow 1 \quad (12)$$

Formulas (11) and (12) were verified by table in article 16. The meaning of (11) and (12) is that for certain angles the sine and tangent may be replaced by the angle and the work of solving certain problems greatly simplified. The values of such angles depend entirely upon the degree of accuracy desired.

If work correct to third place is sufficient, then for $\theta = 5^\circ$ or any angle less, we may substitute the circular measure of the angle for the sine or tangent. For example if you will refer to table A you will find

$$\begin{array}{rcl} \sin 5^\circ & = & .08716 = .087 \\ 5 & .0872 & = .087 \text{ (in circular measure)} \\ \tan 5^\circ & .08749 & .087 \end{array}$$

This shows the correctness of the preceding statement.

EXAMPLES FOR PRACTICE.

1. Determine the greatest value of θ , for which we may replace θ by $\sin \theta$ and $\tan \theta$, in order that the work may be correct to four places. Consult a table of natural functions.

2. Find the angle subtended by a man 5 feet high at a distance of 2,000 feet.

Solution: From Fig. 11a $\tan \theta = \frac{5}{2000} = \frac{1}{400}$

∠ Since one's eye tells him that the angle is less than 5° we may use (12) and replace $\tan \theta$ by θ .

Then $\theta = \frac{1}{400}$ (circular measure). In order to express θ in degrees we apply the principles of Art. 7.

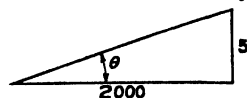


Fig. 11a

$$\angle \theta = \frac{1}{400} \times \frac{180}{\pi} = \frac{1}{400} \times \frac{7}{22} \times 180 \times 60 = \frac{189'}{22} = 8'.59$$

($\pi = \frac{22}{7}$)

3. A tower is 100 feet high. At what distance is a man from the tower when the subtended angle is $1'$?

$$\text{Solution: } \tan 1' = 1' = \frac{\pi}{180} \text{ radians } \angle \frac{100}{x} = \frac{\pi}{180}$$

$$x = \frac{100 \times 180}{22} \times \frac{7}{60} = 5727.272 \text{ ft.}$$

4. A circular target 6 ft. high subtends an angle of $8'$. How far away is the target?

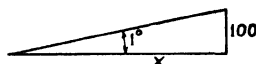


Fig. 11b

5. A railway track rises 90 ft. to the mile. Find the angle of inclination of the track to the horizontal.

6. A railway track has a 1 per cent grade for a certain distance. Find the inclination of the track to the horizontal.

Solution: The per cent of a grade is the ratio of the number of feet rise per mile to the number of feet in a mile. Then for a 1 per cent grade we would have 1 per cent of $5,280 = 52.8$ ft. rise per mile. In Fig. 12, $AB = 5,280$ ft. and $BC = 52.8$ ft.

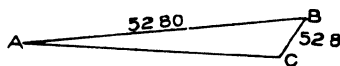


Fig. 12.

$$\angle BAC \text{ in radians} = \frac{52.8}{5280} \quad \angle BAC \text{ in degrees} = \frac{52.8}{5280} \times \frac{180}{\pi} = \frac{1.8}{\pi} = .573^\circ = 34' 22''.8.$$

7. Three thousand feet of a railway track is on a grade which is inclined to the horizontal at an angle of $38'$. Find the per cent of the grade.

There are many problems offered for solution in which two sides of the triangle are practically equal to each other, and each very great compared with the third side. Such a condition is shown in the accompanying Fig. 13 where AB and AC are con

sidered equal. They may be taken for the radii of an arc of which BC the third side of the triangle is a chord. Since the arc is very small, no appreciable error would arise if chord BC is put equal to arc BC . Then

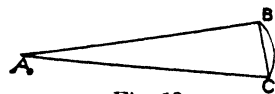


Fig. 13.

$$\angle BAC \text{ in radians} = \frac{\text{arc } BC}{AC} = \frac{\text{chord } BC}{AC} \quad (13)$$

Then when two of the quantities in (13) are known the third may be found.

8. Telescopes at the ends of a line 225 feet long on the deck of a ship are turned upon a distant ship. The lines of sight are found to make angles of $89^\circ 5'$ and $89^\circ 25'$ respectively with the base line. Find the distance between the ships.

Solution: In Fig. 12, let B and C be the positions of the telescopes and A the distant ship. $BC = 225$ ft. and $\angle BAC = 1^\circ 30'$. From (13) $1.5 = \frac{1.5 \times \pi}{180} \text{ radians} = \frac{225}{AB}$

$$\therefore AB = \frac{225}{1.5 \times \frac{\pi}{180}} = 8594.3 \text{ ft.} \quad 1.63 \text{ miles.} \quad (\text{Without logarithms.})$$

9. The length of the line on the deck of a ship equals 200 ft. The angles made by the lines of sight of two telescopes at the ends of the base line trained on a distant fort are $89^\circ 15'$, $88^\circ 55'$ respectively. Find the distance from the ship to the fort.

10. The diameter of the moon subtends an angle of $31'$ at the earth and the distance of the moon from the earth is approximately 240,000 miles. Find the diameter of the moon in miles.

11. The triangle LMN (Fig. 13a) represents the section of two planes. The end L is free to move on the line KM , but N is connected with M by means of a screw that lifts N vertically. In one complete turn of the screw N is lifted through .0403 centimeters. Through what angle will LN be lifted by .113 of a turn.

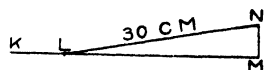


Fig. 13a.

$$\therefore \angle MLN \text{ in radians} = \frac{MN}{LM} = \frac{.0403 \times .113}{30}$$

$$\therefore \angle MLN \text{ in degrees} = \frac{.0403 \times .113}{30} \times \frac{180}{\pi} = 31''.32$$

12. Find the angles if the screw is turned through .175 and .267 of a revolution.

The principles involved in problems 11 and 12 are applicable when calibrating a level; that is, when we wish to determine through what angle one end of the level must be raised to cause the bubble to move between divisions on the glass of the level.

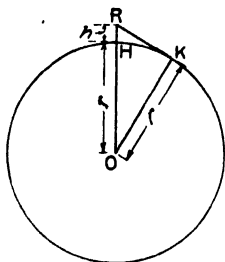


Fig. 14.

17. To find the distance of the visible horizon. Let O be the center of the earth and RH a tower on the surface. It is desired to know the greatest distance from which the top of the tower may be seen. Let HK be this distance. Draw RK , OK and OR . $\angle OKR = 90^\circ$, from geometry. Let $RH = h$. Then

$$RK^2 = (r + h)^2 - r^2 = r^2 + 2hr + h^2 - r^2 = 2hr + h^2$$

Now $r = 3,960$ miles, and h the height of the tower in feet, is $\frac{h}{5280}$ in miles. Suppose $h = 100$ ft. In miles it is $\frac{10}{5280} = .019$. $h^2 = .00036$. This shows that if we merely wish a result correct to first or second decimal places, h^2 may be neglected.

$$\therefore RK = \sqrt{\frac{2 \times 3960 \times h}{5280}} = 1 \frac{1}{2} h \text{ in miles.} \quad (K)$$

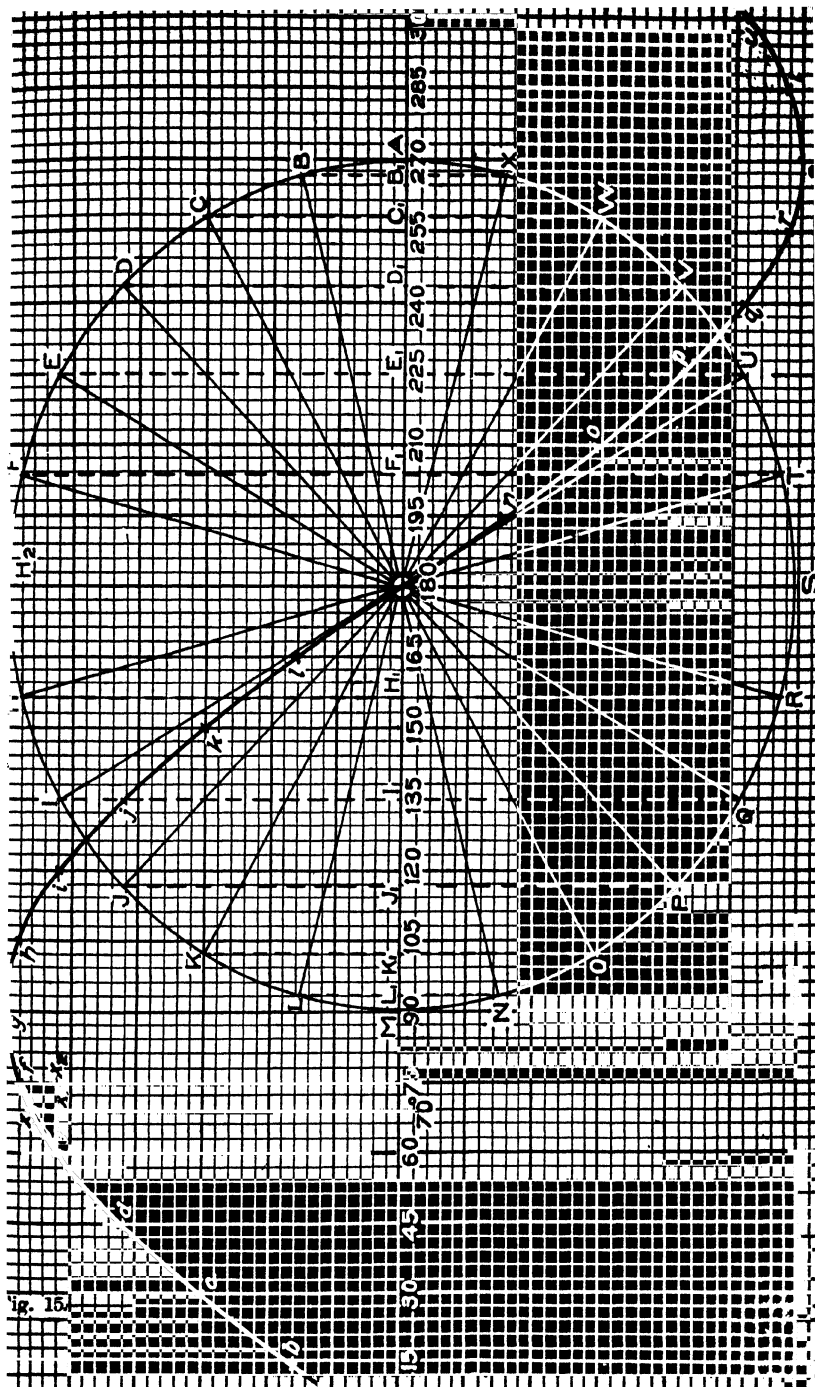
1. Suppose the light in a lighthouse is 75 feet above sea level. How far at sea may it be seen? Here $h = 75$. The distance $= 1 \frac{1}{2} \times 75 = 11$ miles approximately.

2. Find the height of a lighthouse that may first be seen at a distance of 9 miles.

3. From a lighthouse tower which is 50 feet high, the top of a steamer funnel, which is 25 feet above water, is just visible. Find the distance of the steamer.

PLOT OF SINE CURVE.

18. With a radius of thirty spaces, describe on a sheet of co-ordinate paper, a circle with the center O at the intersection of two heavy lines, as shown in Fig. 15. The horizontal line A_1Z is chosen as the axis of abscissas, and the initial line. By means of the protractor construct angles every 15 degrees, so that



$\angle AOB = 15^\circ$, $\angle AOC = 30^\circ$, $\angle AOD = 45^\circ$. . . to 360° . From B, C, D, E, . . the points where the terminal lines of the angles cut the circumference, drop perpendiculars BB_1 , CC_1 , DD_1 . . ., to the initial line. From the point A_1 , chosen as the origin of co-ordinates, plot the angles every 15° , using as a unit, one space for three degrees.

We saw in Article 10, that when the radius of the circle was adopted as the unit of measurement, the perpendiculars, dropped from where the terminal line cuts the circumference to the initial line, represented the sines of the angles.

Then in Fig. 15, $B_1B = \sin 15^\circ$, $C_1C = \sin 30^\circ$, $D_1D = \sin 45^\circ$, $K_1K = \sin 150^\circ$.

Follow the horizontal line through B until it intersects the vertical line through 15° on A_1Z at b . The co-ordinates of b are $(15^\circ, \sin 15^\circ)$. Follow the line through C, until it intersects the vertical line through 30° , locating C. The co-ordinates of c are $(30^\circ, \sin 30^\circ)$. In the same way determine d, e, f, g, h, i - with co-ordinates, $(45^\circ, \sin 45^\circ)$, $(60^\circ, \sin 60^\circ)$, $(75^\circ, \sin 75^\circ)$, $(90^\circ, \sin 90^\circ)$

Draw a smooth curve from A_1 through the points b, c, d, e, f, g $-g, r, s, t$ to z . We then have the *sinusoid* or *sine curve*.

The line A_1Z might have been taken equal in length to the circumference of the circle and then both abscissas and ordinates would have been plotted to the same scale. But it was more convenient to plot the angles as shown in the figure.

Suppose we wish to determine $\sin 40^\circ$ from the curve. Now $40^\circ = 13\frac{1}{3}$ spaces. Measure the ordinate for $13\frac{1}{3}$ spaces. We find it to be 1.5208 inches. The radius was taken as 30 spaces = 2.375 inches.

$$\text{Then} \quad \sin 40 = \frac{1.5208}{2.375} = .64034$$

From a table of natural sines,

$$\sin 40 = .64279.$$

A comparison of the two results shows that if we only care to have our work correct to the hundredths place the results obtained from the curve by careful measurements are sufficiently accurate. The student must remember that the ordinate does not

represent the sine of the angle until expressed in terms of the radius. That is, until it is divided by the radius.

The student is asked to measure the ordinates for the angle given in the following table, compute the sines of those angles and tabulate them. Then find the sines of the same angles from the tables of sines and tabulate.

0	Sines from Curve	Sines from Tables
12		
68		
82		
126		
165		
220		
260		
300		
325		
335		
355		

A comparison of the two columns will give an idea of the accuracy of the curve. It does not require a proof for one to see that the more points we plot—that is, the closer the angles are taken together—the more accurate the curves will be.

INTERPOLATION.

19. When we know the functions for two numbers, and wish to find the same function for a number between the two given numbers, we resort to interpolation. In this we assume that the function changes uniformly with the angle. That is, equal changes in the angle produces equal changes in the function. For example, the ordinates for $60'$ and $75'$ measured on Fig. 15, are 2.05 and 2.28 inches respectively. In order to find the ordinate for $70'$ we proceed as follows :

$$75^\circ - 60^\circ = 15, \quad 70^\circ - 60^\circ = 10'$$

$$10^\circ = \frac{10}{15} \text{ or } \frac{2}{3} \text{ of the distance from } 60^\circ \text{ to } 75^\circ.$$

$2.28 - 2.05 = .23$. Since we assume that the change in the ordinate is proportional to the angle, the change from 60° to 70° would be $\frac{2}{3}$ of .23 = .15333 . . .

Then ordinate for $70^\circ = 2.05 + .153 = 2.203$. . .

Join e and f' of Fig. 15, and draw $e x_2$ parallel to A Z. Let x be

the point where the ordinate for 70° cuts $e f$. From geometry, triangles $e x x_1$ and $e f x_2$ are similar. Then

$$x x_1 : f x_2 :: e x_1 : e x_2.$$

Then $x x_1$ must be the increase in the ordinate, while passing from $60'$ to $70'$. That is, $x x_1 = .1533$.

Then the line from x to 70° with length 2.203 is the interpolated ordinate for $70'$. The line from x_0 to 70° is the actual ordinate, which measured is 2.234 inches. The error is $2.234 - 2.203 = .031$ inches, which equals the segment $x x_0$.

Now from geometry, the smaller the arc the more nearly does it coincide with its chord. Then the nearer the points between which we interpolate the shorter is the segment $x x_0$. That is, the error becomes smaller. Good tables have values inserted for every minute.

PLOT OF COSINE CURVE.

20. In Fig. 15, let $S G$ be chosen as an initial line. Then $\angle H O G = 15^\circ$, $\angle I O G = 30^\circ$, $\angle J O G = 45^\circ$, etc. From article 10, if we drop a perpendicular from H to $O G$, then $O H_2$ when expressed in terms of the radius will represent the cosine of $\angle H O G$. But $H_1 H = O H_2$. Then we may write $\cos 15^\circ = O H_2$, $\cos 30^\circ = I_1 I$, $\cos 45^\circ = J_1 J$, $\cos 60^\circ = K_1 K$, $\cos 75^\circ = L_1 L$, $\cos 90^\circ = 0$, etc. If we erect perpendiculars at O , 15° , 30° , . . . on line $A Z$ of Fig. 15 equal to $H_1 H$, $I_1 I$, $J_1 J$, $K_1 K$, . . . as we did for the sine and draw a curve through the points determined we will have the cosine curve. The co-ordinates of the points plotted will be $(0, \cos 0)$, $(15^\circ, \cos 15^\circ)$, $(30^\circ, \cos 30^\circ)$ etc.

Exercise. Plot the sine and cosine curves when the radius of the circle is 35 spaces.

21. The sine and cosine curves might have been plotted by finding the values of $\sin x$ and $\cosine x$ from the table of natural functions. The absolute values taken from the tables can then be reduced to any desired unit of measurement.

Exercise. Plot the curve with the equation $y = \sin 2x$. Here when $x = 15^\circ$, $y = \sin 30^\circ$, $x = 30^\circ$, $y = \sin 60^\circ$, etc. The co-ordinates of the points to be plotted are $(0, 0)$, $(15^\circ, \sin 30^\circ)$, $(30^\circ, \sin 60^\circ)$, $(45^\circ, \sin 90^\circ)$, etc.

2. On the same sheet of co-ordinate paper, using the same

units, plot the curves whose equations are $y = 2 \sin 2x$, and $y = \sin 3x$. Plot x every 15° .

3. Plot the curve with equation $y = x \sin x$.

Solution: If we assume values for x in degrees, say $x = 15^\circ$, then $y = 15^\circ \sin 15^\circ$. Before y is shown, we may reduce 15° to circular measure. Then $y = \frac{15}{57.3} \sin 15^\circ$.

If we had assumed linear values for x , say $x = \frac{3}{4}$, then $y = \frac{3}{4} \sin \frac{3}{4}$; here we may reduce $\frac{3}{4}$ radians to degrees before y is completely known.

$$\frac{3}{4} \text{ radians} = \frac{3}{4} \times 57.3 = \frac{171.9}{4} = 42.975$$

Or $y = \frac{3}{4} \sin (42.975)$.

The curve may be plotted by either of the methods described, but the latter is a preferable one, as x is more easily plotted in linear units than in degree. For the above curve assume $x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3$, etc.

4. Plot the curve whose equation is $y = 2 \sin (\frac{1}{2} x + 1)$.

5. Plot the curve $y = \tan x$. Also $y = \cot x$.

6. Construct very accurately three right angled triangles, numbered I, II, and III. Measure the lengths of the three sides. Calculate $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$ for each triangle, where x represents one of the acute angles. Tabulate the results as shown below. Compute each result correct to third place.

$\triangle$	$\sin a$	$\cos a$	$\sin^2 a + \cos^2 a$	Should be,
I				1
II				1
III				1

$\triangle$	$\tan a$	$\sec a$	$1 + \tan^2 a$	$\sec^2 a$
I				
II				
III				

$\triangle$	$\cot a$	$\csc a$	$1 + \cot^2 a$	$\operatorname{cosec}^2 a$
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22. Fundamental Formulas. In Part I, Arts. 29 and 30, the following formulas were derived:

$$(14). \quad \sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$(15). \quad \sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$(16). \quad \cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$(17). \quad \cos(x - y) = \cos x \cos y + \sin x \sin y$$

As stated in Part I, article 31, these formulas were proved true only when $x < 90^\circ$, $y < 90^\circ$ and $x + y < 90^\circ$. We will now prove formula (14) true for angles in any quadrant.

Let $x < 90^\circ$, $y < 90^\circ$ and $x + y > 90^\circ < 180^\circ$. Assume that $x = 90^\circ - \theta_1$, and $y = 90^\circ - \theta_2$. Then $x + y = 180^\circ - (\theta_1 + \theta_2)$

$$\theta_1 + \theta_2 = 180^\circ - (x + y).$$

But $x + y > 90^\circ$. Then $\theta_1 + \theta_2 < 90^\circ$.

$$\begin{aligned} \sin(x + y) &= \sin(90^\circ - \theta_1 + 90^\circ - \theta_2) \\ &= \sin(180^\circ - (\theta_1 + \theta_2)) = \sin(\theta_1 + \theta_2). \quad (a) \end{aligned}$$

Since $\theta_1 + \theta_2 < 90^\circ$, we may apply [9] Part I, and

$$\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2$$

$$\text{Now} \quad \sin \theta_1 = \sin(90^\circ - x) = \cos x$$

$$\cos \theta_1 = \cos(90^\circ - x) = \sin x$$

$$\sin \theta_2 = \sin(90^\circ - y) = \cos y$$

$$\cos \theta_2 = \cos(90^\circ - y) = \sin y$$

Substituting in equation (a) we have

$$\begin{aligned} \sin(x + y) &= \sin(\theta_1 + \theta_2) = \\ &= \cos x \sin y + \sin x \cos y \quad \text{Q. E. D.} \quad (b) \end{aligned}$$

Now assume that x and y are each in the second quadrant. Then $x = 180^\circ - \theta_1$, $y = 180^\circ - \theta_2$ where θ_1 and θ_2 are each acute.

$$\text{Then} \quad \theta_1 = 180^\circ - x, \quad \theta_2 = 180^\circ - y$$

$$\theta_1 + \theta_2 = 180^\circ - x + 180^\circ - y = 360^\circ - (x + y). \quad (c)$$

Since x and y are each in the second quadrant, $180^\circ < x + y < 360^\circ$. Then from (c)

$$\theta_1 + \theta_2 < 180^\circ.$$

$\sin(x + y) = \sin(180^\circ - \theta_1 + 180^\circ - \theta_2) = \sin(360^\circ - (\theta_1 + \theta_2)) = -\sin(\theta_1 + \theta_2)$ from law expressed in article 22. We take $\sin(\theta_1 + \theta_2)$ because 360 is an even multiple of 90° . We prefix the minus sign because $x + y$ is in either the third or fourth quadrants and the sine is minus.

In (b) we proved that when $x + y$ or $\theta_1 + \theta_2$ was less than 180° , (9) was true. Then we may write

$$-\sin(\theta_1 + \theta_2) = -\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2 \quad (\text{d})$$

But $\sin \theta_1 = \sin(180^\circ - x) = \sin x$

$$\cos \theta_1 = \cos(180^\circ - x) = -\cos x \quad (x \text{ in 2nd quadrant})$$

$$\sin \theta_2 = \sin(180^\circ - y) = \sin y$$

$$\cos \theta_2 = \cos(180^\circ - y) = -\cos y$$

Substituting in (d),

$$\begin{aligned} -\sin(\theta_1 + \theta_2) &= -(\sin x)(-\cos y) - (-\cos x)(\sin y) \\ &= \sin x \cos y + \cos x \sin y \end{aligned}$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y.$$

In the same way we may extend this proof to angles in the third and fourth quadrants. That is, formula (14) is true for angles in any of the four quadrants.

Formulas (15), (16) and (17) may all be generalized by making similar assumptions to those in (14). The student is asked to prove at least one of them, as he will find the work exceedingly helpful.

EXAMPLES FOR PRACTICE.

1. Show that formula (16) is true when x is in the fourth and y in second quadrant.

2. Show that formula (15) holds when x and y are both in the third quadrant.

3. Show that formula (17) holds when x is in second and y in the third quadrant.

4. Find $\sin 75^\circ$. Hint: $75^\circ = 45^\circ + 30^\circ$

$$\begin{aligned} \text{Then } \sin 75^\circ &= \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ \\ &= \frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}(\sqrt{3} + 1) \end{aligned}$$

5. Find $\cos 15^\circ$ and $\tan 15^\circ$.

6. Given $\sin \alpha = \frac{4}{5}$, $\cos \beta = \frac{3}{5}$, $\alpha < 90^\circ$, $\beta < 90^\circ$

Find $\cos(\alpha + \beta)$; $\sin(\alpha - \beta)$

7. Given $\cos \alpha = -\frac{4}{5}$, $\cos \beta = \frac{3}{5}$

α in III, β in II quadrants. Find $\sin(\alpha + \beta)$, $\cos(\alpha - \beta)$, $\tan(\alpha + \beta)$.

8. Given $\tan \gamma = -\frac{1}{2}$, $\cos \alpha = -\frac{4}{5}$

γ in IV; α in II quadrants. Find $\tan(\gamma + \alpha)$ and $\tan(\gamma - \alpha)$.

GEOMETRICAL PROOF OF FORMULAS 19, 20 AND 24, IN PART I.

23. Draw the circular arc, radius OE as shown in Fig. 16. Bisect $\angle EOA$ by OB . Draw BD and $AC \perp$ to OE . Then $\angle EOA = 2x$, $\angle EOB = \angle BOA = \angle DBE = x$. Now $AC = 2BD$, since $AE = 2BE$ and $\triangle BOE$ and $\triangle ACE$ are similar.

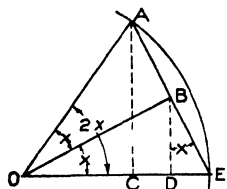


Fig. 16.

$$\sin 2x = \frac{AC}{OA} = \frac{2BD}{OA} = \frac{2BD}{OB} \times \frac{OB}{OA}$$

But

$$\frac{BD}{OB} = \sin x, \text{ from } \triangle DOB, \text{ and } \frac{OB}{OA} = \cos x, \text{ from } \triangle OBA.$$

Substituting these results in $\sin 2x$ we have ($\sin 2x = 2 \sin x \cos x$) (18)

$$\text{Also } \cos 2x = \frac{OC}{OA} = \frac{OD}{OA} - \frac{CD}{OA} = \frac{OD}{OA} - \frac{ED}{OA}$$

$$\text{Since } ED = CD. \text{ Then } \cos 2x = \frac{OD}{OB} \cdot \frac{OB}{OA} = \frac{ED}{EB} \cdot \frac{EB}{OE},$$

since $OE = OA$

$$\text{But } \frac{OD}{OB} = \cos x; \frac{OB}{OA} = \cos x$$

$$\frac{ED}{EB} = \sin x, \frac{EB}{OE} = \sin x$$

$$\therefore \cos 2x = \cos^2 x - \sin^2 x \quad (19)$$

1. Given $\sin x = -\frac{1}{5}$, x in III quadrant.

Find $\sin 2x$, $\cos 2x$, $\tan 2x$.

2. $\tan x = -\frac{1}{2}$, x in IV quad. Find $\sin 2x$ and $\cos 2x$.

3. $\cot x = \frac{4}{3}$, x in III quad. Find $\cos 2x$ and $\tan 2x$.

4. Prove $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$, from Fig. 16.

HALF ANGLE IN TERMS OF THE ANGLE.

24. Let $\angle BAK = x$, be the given angle as shown in Fig. 17. On AB as diameter construct a semicircle. Draw KB , OK and draw $KL \perp$ to AB .

Then $\angle KOB = 2x$

$\sin x = \frac{KL}{AK}$. But AKB is a right angled triangle.

$\therefore KL = \sqrt{AL \cdot LB}$ and $AK = \sqrt{AB \cdot AL}$ from geometry,

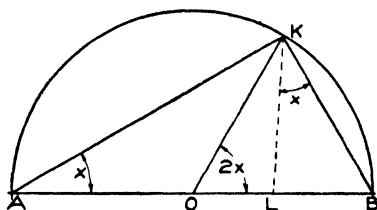


Fig. 17.

$$\begin{aligned} \text{Then } \sin x &= \frac{\sqrt{AL \cdot LB}}{\sqrt{AL \cdot AB}} = \sqrt{\frac{LB}{AB}} = \sqrt{\frac{OB - OL}{2OB}} \\ &= \sqrt{\frac{OB}{2OB} - \frac{OL}{2OB}} = \sqrt{\frac{1}{2} - \frac{\cos 2x}{2}}. \quad (20) \end{aligned}$$

since $OK = OB$ and $\cos 2x = \frac{OL}{OK} = \frac{OL}{OB}$

$$\begin{aligned} \cos x &= \frac{AL}{AK} = \frac{AL}{\sqrt{AB \cdot AL}} = \sqrt{\frac{AL}{AB}} = \sqrt{\frac{AO + OL}{2AO}} \\ &= \sqrt{\frac{1}{2} + \frac{OL}{2OK}} = \sqrt{\frac{1}{2} + \frac{\cos 2x}{2}}. \quad (21) \end{aligned}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\sqrt{\frac{1}{2} - \frac{\cos 2x}{2}}}{\sqrt{\frac{1}{2} + \frac{\cos 2x}{2}}} = \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}}. \quad (22)$$

By rationalizing first the numerator and then the denominator of (22) show that

$$\tan x = \frac{1 - \cos 2x}{\sin 2x} = \frac{\sin 2x}{1 + \cos 2x} \quad (23)$$

EXAMPLES FOR PRACTICE.

1. Given the functions of 30° ; find $\sin 15^\circ$, $\cos 15^\circ$ and $\tan 15^\circ$.
2. Given the functions of 45° ; find $\sin 22\frac{1}{2}^\circ$, $\cos 22\frac{1}{2}^\circ$, $\tan 22\frac{1}{2}^\circ$.

EXPERIMENTAL PROBLEMS.

25. In the following problems compute arithmetically the value of the right hand side of the formula, and compare with the left hand side either from the tables or computed.

1. $\sin(50^\circ + 34^\circ) = \sin 50^\circ \cos 34^\circ + \cos 50^\circ \sin 34^\circ$.
Represent the right hand side by $x + y$.

$\log \sin 34^\circ = 9.74756,$ $\log \cos 50^\circ = 9.80807.$ $\log y = 9.55563$ $y = .35944$ $x = .63507$ $x + y = .99451$	$\log \cos 34^\circ = 9.91857$ $\log \sin 50^\circ = 9.88425$ $\log x = 9.80282$ $x = .63507$
--	---

But $\sin(50 + 34) = \sin 84 = .99452$, which shows that the two members balance.

2. $\cos(75^\circ + 43^\circ) = \cos 75^\circ \cos 43^\circ - \sin 75^\circ \sin 43^\circ$
 $= x - y$

$\log \cos 43^\circ = 9.86413,$ $\log \cos 75^\circ = 9.41300,$ $\log x = 9.27713$ $x = .18929$	$\log \sin 43^\circ = 9.83378$ $\log \sin 75^\circ = 9.98494$ $\log y = 9.81872$ $y = .65875$
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$\therefore x - y = -.46946$

But $\cos 118^\circ = \cos(180^\circ - 62^\circ) = -\cos 62^\circ = -.46947$.

3. $\sin(140^\circ - 70^\circ) = \sin 140^\circ \cos 70^\circ - \cos 140^\circ \sin 70^\circ$

4. $\cos(160^\circ - 33^\circ) = \cos 160^\circ \cos 33^\circ + \sin 160^\circ \sin 33^\circ$

5. $\tan 72^\circ = \frac{2 \tan 36^\circ}{1 - \tan^2 36^\circ}$

6. $\sin 132^\circ = 2 \sin 66^\circ \cos 66^\circ$

7. $\cos 150^\circ = \cos^2 75^\circ - \sin^2 75^\circ$
 $= 1 - 2 \sin^2 75^\circ$

8. $\sin 20^\circ = \sqrt{\frac{1 - \cos 40^\circ}{2}}$

9. $\tan 65^\circ = \sqrt{\frac{1 - \cos 130^\circ}{1 + \cos 130^\circ}}$

10. $\cos 81^\circ = 4 \cos^3 27^\circ - 3 \cos 27^\circ$

11. $\cos 160^\circ = 1 - 8 \sin^2 40^\circ + 8 \sin^4 40^\circ$

12. $\sin 50^\circ \cos 30^\circ = \frac{1}{2} \sin(80^\circ) + \frac{1}{2} \sin(20^\circ)$

13. $\cos 60^\circ \sin 80^\circ = \frac{1}{2} \sin 140^\circ - \frac{1}{2} \sin(-20^\circ)$

14. $\cos 110^\circ \cos 70^\circ = \frac{1}{2} \cos(180^\circ) + \frac{1}{2} \cos 40^\circ$

MULTIPLE ANGLES.

26. It is often desirable to express a multiple of an angle θ in terms of θ . A few problems will be inserted here to illustrate the principles.

1 Given $\sin 3x$ to express in terms of $\sin x$.

Solution:

$$\begin{aligned}\sin 3x &= \sin (2x + x) = \sin 2x \cos x + \cos 2x \sin x \\ &= 2 \sin x \cos x \cos x + (1 - 2 \sin^2 x) \sin x \\ &= 2 \sin x (1 - \sin^2 x) + \sin x - 2 \sin^3 x \\ &= 3 \sin x - 4 \sin^3 x\end{aligned}$$

2. Show that $\cos 3x = 4 \cos^3 x - 3 \cos x$

3. Show that $\cos 4x = 1 - 8 \sin^2 x + 8 \sin^4 x$

4. Show that $\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$

26 a. Prove by means of formulas (28), (29), (30), and (31) of Part I the following identities:

1. $\frac{\sin x + \sin y}{\cos x + \cos y} = \tan \frac{1}{2} (x + y)$

Solution:

$$\begin{aligned}\sin x + \sin y &= 2 \sin \frac{1}{2} (x + y) \cos \frac{1}{2} (x - y) \\ \cos x + \cos y &= 2 \cos \frac{1}{2} (x + y) \cos \frac{1}{2} (x - y)\end{aligned}$$

Then

$$\begin{aligned}\frac{\sin x + \sin y}{\cos x + \cos y} &= \frac{2 \sin \frac{1}{2} (x + y) \cos \frac{1}{2} (x - y)}{2 \cos \frac{1}{2} (x + y) \cos \frac{1}{2} (x - y)} \\ &= \frac{\sin \frac{1}{2} (x + y)}{\cos \frac{1}{2} (x + y)} = \tan \frac{1}{2} (x + y)\end{aligned}$$

2. $\frac{\sin x - \sin y}{\cos y + \cos x} = \tan \frac{1}{2} (x - y)$

3. $\frac{\sin x - \sin y}{\cos y - \cos x} = \cot \frac{1}{2} (x + y)$

4. $\frac{\sin 50^\circ - \sin 10^\circ}{\cos 50^\circ - \cos 10^\circ} = -1/3$

5. $\sin 50^\circ - \sin 70^\circ + \sin 10^\circ = 0$

6. $\frac{\sin 60^\circ - \sin 20^\circ}{\cos 60^\circ + \cos 20^\circ} = \tan 20^\circ$

CIRCULAR OR INVERSE TRIGONOMETRIC FUNCTION.

27. The expression $\sin y = x$ is read, y is an angle whose sine is x . Another way of writing this relation is $y = \sin^{-1} x$, read $y = \text{arc sine } x$, or anti-sine x , or y is an angle whose sine is x .

This way of expressing a function is quite useful in the advanced mathematics, as we are able to express the angle definitely in terms of some function of the angle.

The symbol (-1) must not be taken for a negative exponent. For example, $(\sin x)^{-1} = \frac{1}{\sin x}$ and has an entirely different meaning.

Suppose we are given $y = \sin^{-1}(\frac{1}{2})$ to find y . The angle whose sine is $\frac{1}{2}$ is 30°

$$\therefore y = 30^\circ \text{ or } \sin^{-1} \frac{1}{2} = 30^\circ$$

1. Find y in each of the following expressions:

$$y = \sin^{-1}(1) \quad y = \sin^{-1}\left(\frac{1}{2}\right) \quad y = \tan^{-1}(\infty)$$

$$y = \cos^{-1}(0) \quad y = \cos^{-1}(.2) \quad y = \cos^{-1}\left(\frac{1}{2}\right)$$

$$y = \tan^{-1}(-1) \quad y = \tan^{-1}\left(\frac{1}{3}\right) \quad y = \cos^{-1}(0)$$

2. Given $y = \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}$ to find y

Assume that $x = \tan^{-1} \frac{1}{2}$, and $z = \tan^{-1} \frac{1}{3}$

Then by definition, $\tan x = \frac{1}{2}$, $\tan z = \frac{1}{3}$

The above angles are supposed to be in the first quadrant, and therefore y is known when one function is known. Let us find $\tan y$.

$$\tan y = \tan(x + z) = \frac{\tan x + \tan z}{1 - \tan x \tan z}$$

Substitute for $\tan x$ and $\tan z$

$$\tan y = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1$$

..

$$y = 45^\circ \text{ or } 225^\circ$$

3. Find $\sin(\tan^{-1} 1' 3'')$; $\tan(\sin^{-1} \frac{1}{2})$; $\cot(\tan^{-1} \frac{5}{12})$; $\cos(\sin^{-1} 0)$.

4. Prove that $\sin^{-1}\left(-\frac{1}{2}\right) = \cos^{-1}\left(\frac{1}{2}\right)$

5. Prove that $\cos(2 \tan^{-1} y) = \frac{1 - y^2}{1 + y^2}$

6. Prove $\sin^{-1} \frac{1}{\sqrt{5}} + \sin^{-1} \frac{1}{\sqrt{10}} = \frac{\pi}{4}$
7. Prove that $2 \tan^{-1} \frac{1}{2} + \sec^{-1} \left(\frac{5}{4}\right) = \frac{\pi}{2}$
8. Plot the curves $x = \sin^{-1} y$; $x = \cos^{-1} y$.
9. $\sin (\sin^{-1} \frac{1}{2}) = ?$ $\cos (\tan^{-1} 1) = ?$ $\tan (\sin^{-1} \frac{1}{2} \sqrt{3}) = ?$
10. Show that $\sin (2 \cos^{-1} x) = 2x \sqrt{1-x^2}$
11. Show that $\cos (2 \sin^{-1} x) = 1 - 2x^2$
12. Show that $\tan (2 \tan^{-1} x) = \frac{2x}{1-x^2}$
13. Show that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

28. In the following problems, transform one member of the equation into the other, by applying formulas previously established

$$1. \sin \frac{1}{2} A = \sqrt{\frac{1 + \text{Versin } A}{2}}$$

Versin $A = 1 - \cos A$ by definition

Then $\sqrt{\frac{1 + \text{Versin } A}{2}} = \sqrt{\frac{1 + 1 - \cos A}{2}}$

But $\cos A = 1 - 2 \sin^2 \frac{1}{2} A$

or $\frac{1 + \cos A}{2} = \sin^2 \frac{1}{2} A$

or $\sqrt{\frac{1 + \cos A}{2}} = \sin \frac{1}{2} A$ Q. E. D

$$2. \cos Z = \frac{1 - \tan^2 \frac{1}{2} Z}{1 + \tan^2 \frac{1}{2} Z}$$

Now $\tan^2 \frac{1}{2} Z = \frac{\sin^2 \frac{1}{2} Z}{\cos^2 \frac{1}{2} Z}$

and $1 + \tan^2 \frac{1}{2} Z = \sec^2 \frac{1}{2} Z$

$$\text{Then } \cos Z = \frac{1 - \frac{\sin^2 \frac{1}{2} Z}{\cos^2 \frac{1}{2} Z}}{\sec^2 \frac{1}{2} Z} = \frac{\cos^2 \frac{1}{2} Z - \sin^2 \frac{1}{2} Z}{\cos^2 \frac{1}{2} Z} = \frac{1}{\cos^2 \frac{1}{2} Z}$$

$$\cos^2 \frac{1}{2} Z - \sin^2 \frac{1}{2} Z$$

But

$$\cos^2 \frac{1}{2} Z - \sin^2 \frac{1}{2} Z = \cos Z$$

 $\therefore$

$$\cos Z = \cos Z.$$

3. Show that $\sin^6 \theta + \cos^6 \theta = 1 - 3 \sin^2 \theta \cos^2 \theta$.

Solution: $\sin^6 \theta + \cos^6 \theta$ may be taken as the sum of two cubes. It may be factored and written

$$\sin^6 \theta + \cos^6 \theta = (\sin^2 \theta + \cos^2 \theta) (\sin^4 \theta - \sin^2 \theta \cos^2 \theta + \cos^4 \theta)$$

But $\sin^2 \theta + \cos^2 \theta = 1$. Adding and subtracting $2 \sin^2 \theta \cos^2 \theta$ in the second factor

$$\begin{aligned} \sin^6 \theta + \cos^6 \theta &= \sin^4 \theta + 2 \sin^2 \theta \cos^2 \theta + \cos^4 \theta - 3 \sin^2 \theta \cos^2 \theta \\ &= (\sin^2 \theta + \cos^2 \theta)^2 - 3 \sin^2 \theta \cos^2 \theta \\ &= 1 - 3 \sin^2 \theta \cos^2 \theta \end{aligned}$$

Q. E. D.

$$4. \sin x = \frac{2 \tan \frac{1}{2} x}{1 + \tan^2 \frac{1}{2} x}$$

$$5. \cot y = \frac{1 + \cos 2y}{\sin 2y}$$

$$6. 1 + \frac{\tan \frac{1}{2} A}{\cot A} = \sec A$$

$$7. \frac{\sec^2 \theta + 2 \sec \theta + 1}{\sec \theta (\sec \theta + \cos \theta - 2)} = \frac{1}{\tan^2 \frac{\theta}{2}}$$

$$8. \frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = 2$$

$$9. \tan^2 \frac{S}{2} + 1 = \frac{2}{\sin S \cot \frac{1}{2} S}$$

$$10. \sin^4 \theta + \cos^4 \theta = 1 - \frac{\sin^2 2\theta}{2}$$

$$11. \cot \frac{1}{2} y + \tan \frac{1}{2} y = \sin y$$

$$12. \sqrt{\frac{\tan \theta}{2 (\tan \theta + \cot 2\theta)}} = \sin \theta$$

$$13. \frac{1 - \cos 2A}{\sin 2A} = \tan A$$

$$14. \frac{1 + \cos B}{2 \cos \frac{1}{2} B} = \cos \frac{1}{2} B$$

$$15. 1 + \cot^2 Z = \frac{2}{\sin 2Z \tan Z}$$

$$16. \frac{\cos A - \sin A}{\cos A + \sin A} = \frac{\cos 2 A}{1 + \sin 2 A}$$

$$17. 1 - 2 (1 - \operatorname{covers} x)^2 + \frac{\tan^4 x}{(1 + \tan^2 x)^2} = \cos^4 x$$

APPLICATIONS OF TRIGONOMETRY.

20. In order to determine a few of the applications of trigonometry to mechanical problems, the following experiment in forces was performed.

Three spring balances were attached to ropes which were connected at O as in Fig. 18. The balances were drawn back and

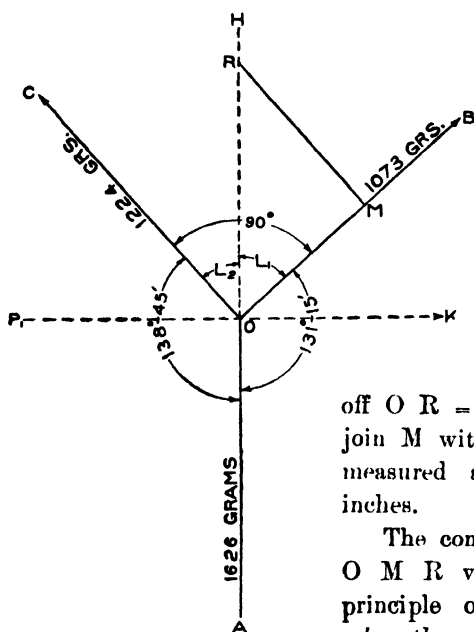


Fig. 18.

fastened at A, B and C. Balance at A read 1,626 grams; B, 1,073 grams; C, 1,224 grams. The unit of measurement was established so that a line one inch long would represent 800 grams. Then A would be 2.03 inches long; B would be 1.34 inches and C 1.53 inches. Lay off O R = 2.03, O M = 1.34, and join M with R. The line M R was measured and found to be 1.53 inches.

The construction of the triangle O M R verified a very important principle of mechanics, *viz.*, that when three forces in the same plane, acting at a point, are in equilib-

rium, a triangle may be constructed with sides parallel to and proportional in length to these forces.

The angles that the forces make with each other, were read by the protractor and found to be as follows:

$$\angle COB = 90^\circ, \angle BOA = 131^\circ 15', \angle COA = 138^\circ 45'.$$

From Fig 18, $\angle ORM = \angle COR = 180^\circ - \angle COA$

Then $\sin \angle O R M = \sin (180^\circ - \angle C O A) = \sin \angle C O A$
 $\angle R M O = \angle C O M.$

$$\sin \angle R M O = \sin \angle C O M.$$

$$\text{also } \angle R O M = 180 - \angle M O A,$$

$$\text{and } \sin \angle R O M = \sin (180 - \angle M O A) = \sin \angle M O A$$

By construction $O M : M R : O R :: O B : O C : O A.$

Now from the law of sines, article 44, Part I,

$$O M : M R : O R :: \sin \angle O R M : \sin \angle R O M : \sin \angle O M R$$

Substituting in this proportion the values of the sines previously derived

$$O B : O C : O A :: \sin \angle A O C : \sin \angle A O M : \sin \angle C O M$$

or

(24)

$$1073 : 1224 : 1626 :: \sin (138^\circ 45') : \sin (131^\circ 15') : \sin 90^\circ$$

Dividing the first three terms by 1626 and replacing the sines by their values

$$.659 : .751 : 1 :: 65935 : 75154 : 1 \quad (\text{A})$$

The identity shown in (a), verifies the second familiar law of mechanics, *viz.*:

If three forces are in equilibrium, they are to each other as the sines of the opposite angles. (B)

30. When two or more forces act from the same point, a single force which would produce the same effect on the body acted upon, is called the resultant of these forces.

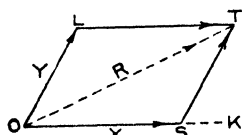


Fig. 19

A familiar principle of physics is, that the resultant of two forces acting at the same point is the diagonal of the parallelogram constructed with sides parallel to and proportional to these forces. (C)

The resultant is the diagonal which acts from the point of application of the original forces. In Fig. 19, let X and Y be two forces acting at O, represented by OS and OL respectively. Now from (C), OT is the resultant = R. By the law of cosines

$$\overline{OT}^2 = \overline{OS}^2 + \overline{OL}^2 - 2 \overline{OS} \cdot \overline{OL} \cos \angle S O L$$

But

$$\angle S O T = 180^\circ - \angle L O S$$

Then

$$\begin{aligned} \cos \angle S O T &= \cos (180^\circ - \angle L O S) \\ &= -\cos \angle L O S \end{aligned}$$

Substituting this value for $-\cos O S T$

$$R^2 = x^2 + y^2 + 2xy \cos \angle L O S \quad (25)$$

EXAMPLES FOR PRACTICE.

1 Find the resultant of O A = 1626 grams; and O B = 1073 grams of figure 18.

Solution: The angle between the forces is $131^\circ 15'$. Then

$$R^2 = 1626^2 + 1073^2 + 2 \cdot 1626 \cdot 1073 \cos (131^\circ 15')$$

$$\log 1626 = 3.211121$$

$$2 \log 1626 = 6.422242$$

$$\therefore \quad 1626^2 = 2643884.1$$

$$\log 1073 = 3.0306$$

$$2 \log 1073 = 6.0612$$

$$\therefore \quad 1073^2 = 1151331.6$$

$$\log 2 = 0.301030$$

$$\log 1626 = 3.211121$$

$$\log 1073 = 3.030600$$

$$\log \cos (131^\circ 15') = 9.819113-10$$

$$\log [2 \cdot 1626 \cdot 1073 \cos (131^\circ 15')] = 6.361864$$

$$\therefore 2 \cdot 1626 \cdot 1073 \cdot \cos (131^\circ 15') = -2300719.6$$

$$2643884.1 \quad 3795215.7$$

$$1151331.6 \quad 2300719.6$$

$$3795215.7 \quad 1494496.1$$

$$R^2 = 2643884.1 \quad 1151331.6 \quad 2300719.6$$

$$R = 1494496.1$$

$$R = 1222.5 \text{ grams.}$$

This result shows the resultant of O A and O B to be equal and opposite to O C = 1224 grams, as we should expect.

2. Find the resultant of O A = 1626 grams, and O C = 1224 grams. The result should be to O B.

3. Find the resultant of O C = 1224 grams and O B = 1073 grams.

31. The converse of (C) is true. That is, any force may be replaced by two forces, which are the sides of a parallelogram with the given force as a diagonal. The two forces are called the **components** of the given force.

The usual way of resolving a force is to resolve it into two perpendicular components. The force with its components then forms a right angled triangle with the given force the hypotenuse and the components the legs. Thus, let RH , Fig. 19a, be the force. Construct the right triangle RHL . RL and $HL = RM$ are the components of RH .

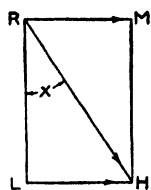


Fig. 19a.

$$\begin{aligned} \text{But} \quad RL &= RH \cos x \\ LH &= RH \sin x \end{aligned} \quad (26)$$

Where x is the angle that the force makes with one of its components. By means of equations (26) we may resolve a force into components along, and perpendiculars to, any given line.

The component along the line is the force multiplied by the cosine of the angle made by the force with the given line. The perpendicular component is the force multiplied by the sine of the same angle. (D)

For example: Resolve the forces OC and OB , Fig. 18, into their components along and perpendicular to the direction GOA . Let x_1 and x_2 be the components of the forces OC and OB along OA , and y_1, y_2 those at right angles.

$$\begin{aligned} \text{From (D)} \quad x_1 &= 1224 \cos (41^\circ 15') = 920.242 \text{ grams} \\ x_2 &= 1073 \cos (48^\circ 45') = 707.482 \text{ grams} \\ \hline x_1 + x_2 &= 1627.724 \end{aligned}$$

But $x_1 + x_2$ is the resultant force of OC and OB along OA , which is seen to be balanced by OA as it should be, since the system is in equilibrium.

$$\begin{aligned} y_1 &= 1224 \sin (41^\circ 15') = 807 \\ y_2 &= 1073 \sin (48^\circ 45') = 806.7 \end{aligned}$$

The forces y_1 and y_2 act in opposite directions and balance each other.

When the angles are known at which forces act, the method for finding the resultant of two forces used in this article is much simpler than that of article 30

EXAMPLES FOR PRACTICE.

1. Resolve $OA = 1626$ and $OB = 1073$, Fig. 18, into their components along and perpendicular to OC . Compare with resultant found in article 30.

2. Resolve $OC = 1224$ and $OA = 1626$ into their components along and perpendicular to OB .

3. Given three forces of 1626 grams 1224 grams and 1073 grams acting at a point. The system is in equilibrium. Find the angles at which the forces act with reference to each other. Compare the results with those given in Fig. 18, as measured by the protractor.

Since the forces are in equilibrium they will form a triangle if properly placed, such as $\triangle ROM$, Fig. 18. The problem requires the solution of $\triangle ROM$ for the three angles. From them, we obtain angles AOB , BOC , COA .

4. Given three forces 40, 35 and 31 oz., acting at a common point. The system is in equilibrium. Find the angles at which the forces act.

Having determined the angles, compute the resultant of the first two forces and show that it is equal to the third.

32. Suppose we have given a right triangle frame BAC as shown in Fig. 20. A weight W is hung at C . It is desired to know the forces exerted by BA , AC and BC , in order that the frame may be held rigid. Since the three forces at C , namely the weight W , and the forces in AC and BC are in equilibrium, by article 30 we may construct a triangle BAC with sides parallel and proportional to these forces. That is BA must be proportional to the weight W .

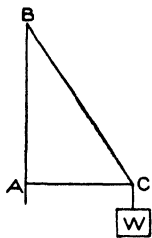


Fig. 20.

In order to illustrate the principle of Fig. 20, the following experiment was performed:

A weight of 10 lb. was hung at C , Fig. 20. The sides of the triangle were measured and found to be $AB = 59.8$ cm., $AC = 34.5$ cm., $BC = 69$ cm. The weight W is fastened to a string BC . A spring balance is attached at C , and a horizontal pull exerted sufficient to lift the string away from AC . The balance read $5\frac{1}{2}$ lb. The balance

was then inserted in the string between B and C so as to sustain the pull and read $11\frac{1}{2}$ lb. From the proportion

$$69 : 59.8 : 34.5 :: 11\frac{1}{2} : 10 : 5\frac{1}{2}.$$

This becomes after reduction

$$1 : .866 : .5 :: 1 : .869 : .5$$

The result shows that B A represents W, and that the lengths of B C and A C are proportional to the forces they exert.

Then, when three forces act at a point, two at right angles with each other, we may construct a right angled triangle with the perpendicular forces as the legs, and the third force as the hypotenuse. The solution of the problem will depend upon the solution of the triangle.

EXAMPLES FOR PRACTICE.

1. Suppose in Fig. 20, A B = 36.7 cm. A C = 34.5, B C = 50.1. With a weight of 10 lb., find the forces in A C and B C
Hint. $50.1 : 36.7 : 34.5 :: x : 10 : y$.

2. A weight of 50 lb. is attached to the rope B C, Fig. 20. The rope B C makes an angle of 30° with A C. Find the tension in the rope and the push on the brace A C.
Hint. A B = 50. $\angle B C A = 30^\circ$ Solve triangle.

3. A car is pulled by a horizontal force of 400 pounds, making an angle of 40° with the track. What is the force tending to make the car move forward?

4. A weight of 100 lb. suspended by a string is drawn aside by a horizontal force until the string makes an angle of 40° with the vertical. What is the intensity of the force, and the tension in the string?

5. A weight of 25 lb. hangs by two strings 3 and 4 feet long respectively which are fastened to the underside of a beam. The points of attachment are 5 feet apart. Find the tension of each string.

6. A boat is fastened by a rope to a point on shore. The wind blows at right angles to the current with a force of 25 lb. The current carries the boat down stream until the pull on the rope is 40 lb. Find the force of the current and the angle that the rope makes with the direction of the current.

33. To find the Height and Distance of an Inaccessible Object on a Horizontal Plane.

Let $H R$, Fig. 21, be the object whose height x and distance $L R$ is desired. Two points K and L in direct line with R are located and $K L$ is measured. The angles $H L R$ and $H K R$ are measured.

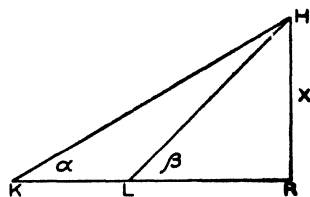


Fig. 21.

By the law of sines

$$K L : L H :: \sin (\beta - \alpha) : \sin \alpha$$

or
$$L H = \frac{K L \sin \alpha}{\sin (\beta - \alpha)}$$

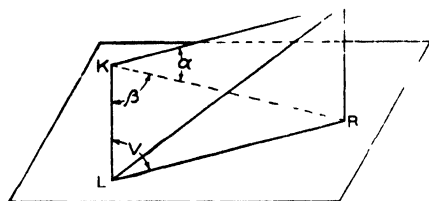


Fig. 22.

Knowing $L H$, $H R$ and $L R$ are easily found by solving the right angled triangle $L H R$.

If K and L cannot be chosen in line with R , choose them as in Fig. 22; read the angles α , β , and γ .

Measure $K L$. Again by the law of sines

$$K R : K L :: \sin \gamma : \sin [180 - (\beta + \gamma)]$$

or
$$K R = \frac{K L \sin \gamma}{\sin (\beta + \gamma)}$$

The height $H R$ may now be found by solving the right triangle $K H R$.

MISCELLANEOUS PROBLEMS.

1. A river 400 feet wide runs at the foot of a lighthouse tower which subtends an angle of $28^{\circ} 25'$ at the edge of the opposite bank; find the height of the tower.

2. At 255 feet from the foot of a flagpole the angle of elevation is $\frac{1}{3}$ of what it is at 75 feet; find the height of the pole.

3. What is the height of a hill whose angle of elevation taken at the bottom was 38° , and 400 feet farther off on a level with the bottom, the angle was 26° ?

4. A flagstaff 35 feet high stands on the top of a cliff, and from a point on the seashore the angles of elevation of the highest

and lowest points of the flagstaff are observed to be $48^{\circ} 13'$, $43^{\circ} 52'$ respectively; find the height of the cliff.

5. From each of two stations north and south of each other, the altitude of a balloon was observed to be 45° . Its bearing with respect to the first station was S. E., and with the second N. E.; the stations are 5,000 feet apart. Find the height of the balloon.

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